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Smoothing Effect for Some Schrödinger Equations

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We study the Cauchy problem for the following Schrödinger equation in $\mathbb{R}^n$ $(n \in \mathbb{N})$:

$$\begin{aligned} i\,\partial_t u + \tfrac{1}{2}\Delta u &= V_1 u + (V_2 * |u|^2)u, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0) &= \phi, \qquad x \in \mathbb{R}^n, \end{aligned} \tag{**}$$

where $V_1 = V_1(x) = \lambda_1 |x|^{-\gamma_1}$, $V_2 = V_2(x) = \sum_{k=2}^{3} \lambda_k |x|^{-\gamma_k}$, $\lambda_k \in \mathbb{R}$ $(1 \leqslant k \leqslant 3)$, $0 < \gamma_1 < \min(2, n/2)$, $0 < \gamma_2, \gamma_3 < \min(2, n)$. We prove the existence, uniqueness, and smoothing effect of global solutions of (**) with ϕ not necessarily in the Sobolev space $H^1(\mathbb{R}^n)$. © 1989 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the following Schrödinger equation in $\mathbb{R}^n$, $n \in \mathbb{N}$:

$$\begin{aligned} i\,\partial_t u + \tfrac{1}{2}\Delta u &= F(u), \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0) &= \phi, \qquad x \in \mathbb{R}^n. \end{aligned} \tag{**}$$

Here the nonlinear interaction term F is written as

$$F(u) = V_1 u + (V_1 * |u|^2)\, u$$

with $V_1 = V_1(x) = \lambda_1 |x|^{-\gamma_1}$, $V_2 = V_2(x) = \sum_{k=2}^{3} \lambda_k |x|^{-\gamma_k}$, $\lambda_k \in \mathbb{R}$ $(1 \leqslant k \leqslant 3)$, $0 < \gamma_1 < \min(2, n/2)$, $0 < \gamma_k < \min(2, n)$, and $*$ denotes the convolution in $\mathbb{R}^n$.

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Our main purpose in this paper is to prove the existence, uniqueness, and smoothing effect of global solutions to the initial value problem (**) with ϕ in the weighted L^2-spaces but not necessarily in the Sobolev space $H^1(\mathbb{R}^n)$. Our formulation of the smoothing properties in space variables is similar to that of A. Jensen [11]. He treats the *linear* case $F(u)=Vu$ with some regularity condition on V which our potential V_1 violates. In the case $\lambda_1=0$, $\lambda_2=\lambda_3$, related results have been obtained by N. Hayashi and Y. Tsutsumi [9] under the additional assumption that $\phi\in H^1(\mathbb{R}^n)$. The proof presented here is based on systematic use of the space-time estimates of the free Schrödinger evolution group $\{U(t);\ t\in\mathbb{R}\}$ with the operators $J=U(t)\,xU(-t)=x+it\nabla$, $J^2=U(t)\,|x|^2\,U(-t)$, and $K=J^2+2t^2L=J^2+2t^2(i\partial_t+\frac{1}{2}\Delta)$.

Throughout the paper we use the following notations:

Notation 1. $\partial_t=\partial/\partial t$, $\partial_k=\partial/\partial x_k$, $\nabla=(\partial_1,...,\partial_n)$, $x=(x_1,...,x_n)$, $\hat{x}=x/|x|$; $A=(\frac{1}{2i})(x\cdot\nabla+\nabla\cdot x)$; $\mathscr{F}$ denotes the Fourier transform defined by $(\mathscr{F}\psi)(\xi)=(2\pi)^{-n/2}\int\psi(x)\exp(-i\xi x)\,dx$, $\xi\in\mathbb{R}^n$; $U=U(t)=\exp(i(t/2)\,\Delta)$, $t\in\mathbb{R}$, $\Delta=\sum_{k=1}^n\partial_k^2=\mathscr{F}^{-1}(-|\xi|^2)\mathscr{F}$; $S=S(t)=\exp(i\,|x|^2/2t)$, $t\in\mathbb{R}\backslash\{0\}$; $J_k=J_k(t)=U(t)\,x_k\,U(-t)$, $J=J(t)=U(t)\,xU(-t)=(J_1,...,J_n)$, $J^2=J^2(t)=U(t)\,|x|^2\,U(-t)$, $t\in\mathbb{R}$; $K=J^2+2t^2L$, $L=i\partial_t+\frac{1}{2}\Delta$; $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$, $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $J^\alpha=J_1^{\alpha_1}\cdots J_n^{\alpha_n}$, $\alpha\in(\mathbb{N}\cup\{0\})^n$, $\partial^0=x^0=J^0=\mathbb{1}$; $\mathscr{S}$ denotes the space of rapidly decreasing C^∞-functions from $\mathbb{R}^n$ to $\mathbb{C}$; L^p denotes the Lebesgue space $L^p(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n)\otimes\mathbb{C}^n$, with the norm $\|\cdot\|_p$, $1\leqslant p\leqslant\infty$; $\|\cdot\|=\|\cdot\|_2$; $(\cdot,\cdot)$ denotes the L^2-scalar product and various anti-dualities; $H_p^{m,s}$ denotes the weighted Sobolev space with the norm $\|\psi\|_{m,s,p}=\|(1+|x|^2)^{s/2}\,(\mathbb{1}-\Delta)^{m/2}\,\psi\|_p$, $m,\ s\in\mathbb{R}$, $1\leqslant p\leqslant\infty$; $\|\cdot\|_{m,s}=\|\cdot\|_{m,s,2}$; $C(I;E)$ (respectively, $C_w(I;E)$) denotes the space of strongly (respectively, weakly) continuous functions from an interval $I\subset\mathbb{R}$ to a Fréchet space E; $C^k(I;E)$ denotes the space of k-times continuously differentiable functions from I to E, $k\in\mathbb{N}$; $L^\theta(I;B)$ (respectively, $L^\theta_{\mathrm{loc}}(I;B)$) denotes the space of measurable functions u from an interval I to a Banach space B such that $\|u(\cdot)\|_B\in L^\theta(I)$ (respectively, $\|u(\cdot)\|_B\in L^\theta_{\mathrm{loc}}(I)$), $1\leqslant\theta\leqslant\infty$; $C_b(I;B)=C(I;B)\cap L^\infty(I;B)$; $L^{q,\theta}(I)=L^\theta(I;L^q)$ with the norm $|||\cdot|||_{q,\theta,I}$; $\delta(q)=n/2-n/q$, $2\leqslant q\leqslant\infty$; $\theta(q)=2/\delta(q)=4q/n(q-2)$, $2<q<\infty$; p' denotes the conjugate exponent to $p\in[1,\infty]$; G_{t_0} denotes the integral operator defined by $(G_{t_0}v)(t)=\int_{t_0}^t U(t-s)\,v(s)\,ds$, $t,\ t_0\in\mathbb{R}$; $G=G_{t_0}$. We remark here that G_{t_0} maps $L^{2,1}(I)+L^{q',\theta(q)'}(I)$ into $C(I;L^2)$ continuously, where $I=[t_0-a, t_0+a]$, $a>0$, and $0\leqslant\delta(q)<1$, although there exist several pairs of function spaces between which G_{t_0} can be regarded as a continuous linear operator (see Lemma 2.7 for details).

Different positive constants might be denoted by the same letter C. If necessary, by $C(*,...,*)$ we denote constants depending on the quantities appearing in parentheses.

The following relations will be freely used in the sequel:

$$\begin{aligned}
J_k(t) &= S(t)(it\partial_k)\, S(-t) = x_k + it\partial_k,\\
J(t) &= S(t)(it\nabla)\, S(-t) = x + it\nabla,\\
J^2(t) &= S(t)(-t^2\Delta)\, S(-t) = |x|^2 - 2tA - t^2\Delta,\\
K &= |x|^2 - 2tA + 2it^2\,\partial_t.
\end{aligned}$$

Finally we define

$$\begin{aligned}
&\Gamma = \{q \in (2, \infty);\ \gamma^0/2 < \delta(q) < \min(1, n/2)\},\\
&\gamma^0 = \max(\gamma_1, \gamma_2, \gamma_3),\\
&\mathscr{X} = C(\mathbb{R}; L^2) \cap \bigcap_{q \in \Gamma} L^{\theta(q)}_{\mathrm{loc}}(\mathbb{R}; L^q),\\
&\mathscr{X}_b = C_b(\mathbb{R}; L^2) \cap \bigcap_{q \in \Gamma} L^{\theta(q)}_{\mathrm{loc}}(\mathbb{R}; L^q).
\end{aligned}$$

With these notations we state our main results.

THEOREM 1. *Let* $\phi \in L^2$. *Then*:

(1) *For any* $q \in \Gamma$ *there exists a unique solution* $u \in C(\mathbb{R}; L^2) \cap L^{\theta(q)}_{\mathrm{loc}}(\mathbb{R}; L^q)$ *of the equation*

$$u = U\phi - iGF(u) \tag{$*$}$$

(*as an identity*) *in* $C(\mathbb{R}; L^2) \cap L^{\theta(q)}_{\mathrm{loc}}(\mathbb{R}; L^q)$.

(2) *Moreover,* $u \in \mathscr{X}_b$ *and* u *satisfies* ($*$) *in* $\mathscr{X}_b$ *and the* L^2*-norm conservation law*

$$\|u(t)\| = \|\phi\|, \qquad t \in \mathbb{R}. \tag{1.1}$$

(3) *If in addition* $\gamma^0 < \min(2, n/2)$, *then* u *satisfies* ($**$) *in* $C_b(\mathbb{R}; H^{-2,0})$.

(4) *If in addition* $\lambda_2 = \lambda_3 = 0$, *then* u *has the representation* $u(t) = \exp(-itH)\phi$, $t \in \mathbb{R}$, *where* $H = -(\frac{1}{2})\Delta + V_1$ *denotes the self-adjoint operator in* L^2 *with the domain* $D(H) = H^{2,0}$.

THEOREM 2. *Let* $\phi \in H^{0,1}$ *and let* u *be the solution of* ($*$). *Then*:

(1) $Ju \in C(\mathbb{R}; L^2)$.

(2) *Let* $\alpha(t) = \|Ju(t)\|^2 + 2t^2(V_1 u(t),\ u(t)) + t^2((V_2 * |u|^2)\, u(t),\ u(t))$.

Then α is absolutely continuous on $\mathbb{R}$ and satisfies the identity

$$\alpha(t)=\alpha(\tau)+4\int_\tau^t s\left\{(\tilde{V}_1 u, u)+\frac{1}{2}((\tilde{V}_2 * |u|^2)u, u)\right\} ds, \qquad t, \tau \in \mathbb{R}, \tag{1.2}$$

where $\tilde{V}_k = V_k + (i/2)[A, V_k] = V_k + \frac{1}{2}(x\cdot\nabla)V_k$, $k=1, 2$.

(3) *We have*

$$u \in C^1\left(\mathbb{R}\backslash\{0\}; \bigcap_{j=0}^{1} H^{j-2,-j}\right) \cap C\left(\mathbb{R}\backslash\{0\}; \bigcap_{j=0}^{1} H^{j,-j}\right), \tag{1.3}$$

and u satisfies (**) *in* $C(\mathbb{R}\backslash\{0\}; \bigcap_{j=0}^{1} H^{j-2,-j})$. *Moreover,*

$$tu \in C\left(\mathbb{R}; \bigcap_{j=0}^{1} H^{j-2,-j}\right), \qquad \lim_{t\to\pm 0} \|tu(t)\|_{1,-1}=0, \tag{1.4}$$

$$\|u(t)\|_{1,-1} \leqslant C(\|\phi\|_{0,1})\cdot(|t|^{-1}+1), \qquad t \in \mathbb{R}\backslash\{0\}. \tag{1.5}$$

If in addition $\lambda_2=\lambda_3=0$, then (1.5) *holds with $C(\|\phi\|_{0,1})$ replaced by $C\cdot\|\phi\|_{0,1}$.*

COROLLARY 1.1. *Let $\phi \in H^{0,1}$ and let $u \in \mathscr{X}_b$ be the solution of* (*). *Then $u \in C(\mathbb{R}\backslash\{0\}; L^p)$ for all p such that $2 \leqslant p \leqslant \infty$ $(n=1)$; $2 \leqslant p < \infty$ $(n=2)$; $2 \leqslant p \leqslant 2n/(n-2)$ $(n \geqslant 3)$.*

THEOREM 3. *Let $n \geqslant 2$. Let $\phi \in H^{0,1}$ and let $u \in \mathscr{X}_b$ be the solution of* (*). *Then $Ju \in \mathscr{X}$ and Ju satisfies the integral equation*

$$Ju = Ux\phi - iGJF(u) \qquad \text{in} \quad \mathscr{X}$$

with $JF(u) = V_1 Ju + (V_2 * |u|^2) Ju + it(\nabla V_1)u + 2i(V_2 * (\bar{u}Ju))u$ *in* $C(\mathbb{R}; L^2) + \bigcap_{q\in\Gamma} L^{\theta(q)'}_{\text{loc}}(\mathbb{R}; L^{q'})$.

THEOREM 4. *Let $\phi \in H^{0,2}$ and let $u \in \mathscr{X}_b$ be the solution of* (*). *Then*:

(1) J^2u, $t^2F(u) \in C(\mathbb{R}; L^2)$.

(2) *Let* $\tilde{K}(u) = J^2u + 2t^2F(u)$. *Then $\tilde{K}(u) \in \mathscr{X}$ and $\tilde{K}(u)$ satisfies*

$$\tilde{K}(u) = U\,|x|^2\,\phi - iG\tilde{F}(u) \qquad \text{in} \quad C(\mathbb{R}; L^2) \tag{1.6}$$

with $\tilde{F}(u) = V_1\tilde{K}(u) + (V_2 * |u|^2)\,\tilde{K}(u) + 2i(V_2 * (\mathrm{Im}(\bar{u}\tilde{K}(u))))u + 4it\tilde{V}_1 u + 4it(V_2 * |u|^2)u$ *in* $C(\mathbb{R}; L^2) \cap \bigcap_{q\in\Gamma} L^{\theta(q)}_{\text{loc}}(\mathbb{R}; L^q)$.

(3) *Let* $\beta(t) = \|\tilde{K}(u)(t)\|^2 - 4t^2(V_2 * (\mathrm{Im}(\bar{u}Ju(t))), \mathrm{Im}(\bar{u}Ju(t)))$. *Then* β *is absolutely continuous on* $\mathbb{R}$ *and satisfies the identity*

$$\begin{aligned}\beta(t) = \beta(\tau) + 8 \int_\tau^t s\{&\mathrm{Re}(\tilde{K}(u), (\tilde{V}_1 + \tilde{V}_2 * |u|^2)u) \\ &- (\tilde{V}_2 * (\mathrm{Im}(\bar{u}Ju)), \mathrm{Im}(\bar{u}Ju))\}\, ds \\ &+ 4 \int_\tau^t (V_2 * (\mathrm{Im}(\bar{u}\tilde{K}u)), \\ &|Ju|^2 + 2s^2(V_1 + V_2 * |u|^2)u)\, ds, \qquad t, \tau \in \mathbb{R}. \end{aligned} \tag{1.7}$$

(4) *We have* $u \in C^1(\mathbb{R}\setminus\{0\}; \bigcap_{j=0}^2 H^{j-2,-j}) \cap C(\mathbb{R}\setminus\{0\}; \bigcap_{j=0}^2 H^{j,-j})$, *and* u *satisfies* (**) *in* $C(\mathbb{R}\setminus\{0\}; \bigcap_{j=0}^2 H^{j-2,-j})$. *Moreover,*

$$t^2 u \in C\left(\mathbb{R}; \bigcap_{j=0}^2 H^{j,-j}\right), \qquad \lim_{t \to \pm 0} \|t^2 u(t)\| = 0, \tag{1.8}$$

$$\tilde{K}(u) = J^2 u + 2t^2(i\partial_t u + \tfrac{1}{2}\Delta u) \quad \text{in} \quad C(\mathbb{R}; L^2), \tag{1.9}$$

$$\begin{aligned}\|u(t)\|_{2,-2} \leqslant C(&\|\phi\|_{2,-2}) \\ &\cdot (|t|^{-2} + |t|^{2/(2-\gamma^0)}), \qquad t \in \mathbb{R}\setminus\{0\}.\end{aligned} \tag{1.10}$$

If in addition $\lambda_2 = \lambda_3 = 0$, *then*

$$\|u(t)\|_{2,-2} \leqslant C \cdot \|\phi\|_{0,2} (|t|^{-2} + 1), \qquad t \in \mathbb{R}\setminus\{0\}. \tag{1.11}$$

COROLLARY 1.2. *Let* $\phi \in H^{0,2}$ *and let* $u \in \mathscr{X}_b$ *be the solution of* (*). *Then* $u \in C(\mathbb{R}\setminus\{0\}; L^p)$ *for all* p *such that* $2 \leqslant p \leqslant \infty$ $(n=3)$; $2 \leqslant p < \infty$ $(n=4)$; $2 \leqslant p \leqslant 2n/(n-4)$ $(n \geqslant 5)$.

Remark 1.1. (1) In the case of *linear* Schrödinger equations with time-dependent potentials $V = V(t, x)$, K. Yajima [15] has obtained sufficient conditions on V such that the equation generates a unique strongly continuous unitary propagator $\{U(t, s)\}$ on L^2. Our potential V_1 is a typical example which satisfies these conditions.

(2) In the case of a *single power* $F(u) = \lambda |u|^{p-1} u$ with $\lambda \in \mathbb{R}$ and $1 < p < 1 + 4/n$, analogous results to parts (1)–(2) of Theorem 1 have been obtained by Y. Tsutsumi [14].

(3) Although the identity (1.2) is a variant of the well-known pseudoconformal coservation law (see, e.g., [2, 4, 5]), all available proofs of the identity impose the additional condition $\phi \in H^{1,0}$.

(4) The identity (1.7) is a variant of that of [8], where only the case $n = 3$, $\lambda_k = \gamma_k = 1$, $1 \leqslant k \leqslant 3$, is considered, with the additional assumption

$\phi \in H^{2,0}$. The result in Corollary 1.2 is a generalization of those of [1, 2, 8] as regards the smoothness properties of solutions.

(5) A. Jensen [11] studied the smoothing properties of the evolution $\exp(-itH)$, where $H = -\Delta + V$ with $V \in H_\infty^{k,0}$, $k \in \mathbb{N}$. It is shown in [11] that $\exp(-itH)$ has the estimate

$$\|\exp(-itH)\|_{\mathscr{L}(H^{0,k};H^{k,-k})} \leqslant C \cdot (|t|^{-k} + |t|^k), \qquad t \in \mathbb{R} \backslash \{0\}.$$

THEOREM 5. *Let $\lambda_1 = 0$. Let $\phi \in H^{0,k}$, $k \in \mathbb{N}$, and let $u \in \mathscr{X}_b$ be the solution of (*). Then $u \in C(\mathbb{R} \backslash \{0\}; \bigcap_{j=0}^k H^{j,-j})$. Moreover, $J^\alpha u \in \mathscr{X}$ for $|\alpha| \leqslant k$.*

COROLLARY 1.3. *Let $\lambda_1 = 0$. Let $\phi \in H^{0,[n/2]+1}$ and $u \in \mathscr{X}_b$ be the solution of (*). Then $u \in C(\mathbb{R} \backslash \{0\}; L^\infty)$.*

When $\lambda_1 = 0$, related results have been obtained by N. Hayashi, K. Nakamitsu, and M. Tsutsumi [7], and N. Hayashi and Y. Tsutsumi [9], with the additional condition $\phi \in H^{1,0}$.

2. PRELIMINARY ESTIMATES

We collect here some preliminary estimates. Througout this section Q_γ denotes the function defined by $Q_\gamma = |x|^{-\gamma}$ for $\gamma > 0$.

LEMMA 2.1 (The Gagliardo–Nirenberg inequality). *Let $1 \leqslant q$, $r \leqslant \infty$. Let j, $m \in \mathbb{N} \cup \{0\}$ satisfy $0 \leqslant j < m$. Let p and a satisfy $1/p = j/n + a(1/r - m/n) + (1-a)/q$; $j/m \leqslant a < 1$ if $m - j - n/r \in \mathbb{N} \cup \{0\}$, $j/m \leqslant a \leqslant 1$ otherwise. Then*

$$\sum_{|\alpha|=j} \|\partial^\alpha \psi\|_p \leqslant C(n, m, j, q, r) \times \sum_{|\beta|=m} \|\partial^\beta \psi\|_r^a \, \|\psi\|_q^{1-a}, \qquad \text{for} \quad \psi \in H_r^{m,0} \cap L^q. \tag{2.1}$$

For Lemma 2.1, see, e.g., A. Friedman [3].

LEMMA 2.2. (1) *Let p, q, and γ satisfy $1 < p < q < \infty$, $0 < \gamma < n$, and $1/q = 1/p - (n-\gamma)/n$. Then*

$$\|Q_\gamma * \psi\|_q \leqslant C(n, p, q) \, \|\psi\|_p, \qquad \text{for} \quad \psi \in L^p. \tag{2.2}$$

(2) *Let $n \geqslant 3$. Then*

$$\|Q_1 \psi\| \leqslant (2/(n-2)) \, \|\nabla \psi\|, \qquad \text{for} \quad \psi \in H^{1,0}, \tag{2.3}$$

$$\|Q_1 \psi\| \leqslant (2/(n-2) \, |t|) \, \|J\psi\|, \qquad \text{for} \quad \psi \in H^{1,1} \text{ and } t \in \mathbb{R} \backslash \{0\}. \tag{2.4}$$

Proof. For (2.2)–(2.3), see, e.g., E. M. Stein [13]. Inequality (2.4) follows from (2.3) and the relation $J = S(it\nabla)S^{-1}$ (see [8]). Q.E.D.

LEMMA 2.3. *Let $m \in \mathbb{N} \cup \{0\}$. Let $V \in H_{\infty}^{m,0}$ be a real function. Then*

$$\sum_{|\alpha| \leqslant m} \|\partial^\alpha(V\psi)\| \leqslant C(m)\, \|V\|_{m,0,\infty}\, \|\psi\|_{m,0}, \qquad \textit{for} \quad \psi \in H^{m,0}, \tag{2.5}$$

$$\sum_{|\alpha| \leqslant m} \|J^\alpha(V\psi)\| \leqslant C(m)\, \|V\|_{m,0,\infty}\, (1+|t|^m) \times \sum_{|\alpha| \leqslant m} \|J^\alpha \psi\|, \qquad \textit{for} \quad \psi \in H^{m,m} \textit{ and } t \in \mathbb{R}. \tag{2.6}$$

Proof. We prove (2.6) only. Since $J^\alpha(V\psi) = S(it\partial)^\alpha\,(VS^{-1}\psi)$, $t \neq 0$, *we have*

$$\sum_{|\alpha| \leqslant m} \|J^\alpha(V\psi)\| \leqslant C(m)\, \|V\|_{m,0,\infty} \sum_{k=0}^{m} |t|^k \sum_{|\beta| = k} \|\partial^\beta S^{-1}\psi\|,$$

so that (2.6) holds for $t \neq 0$. If $t = 0$, (2.6) is clear. Q.E.D.

LEMMA 2.4. *Let $0 < \gamma < \min(2, n/2)$.*

(1) *Let q satisfy $\gamma/2 < \delta(q) < \min(1, n/2)$. Then*

$$\|Q_{\gamma/2}\psi\| \leqslant C \cdot (\|\psi\| + \|\psi\|_q), \qquad \textit{for} \quad \psi \in L^2 \cap L^q, \tag{2.7}$$

$$\|Q_{\gamma/2}\psi\| \leqslant C \cdot (\|\psi\| + |t|^{-\delta}\, \|\psi\|^{1-\delta}\, \|J\psi\|^\delta), \qquad \textit{for} \quad \psi \in H^{1,1} \textit{ and } t \in \mathbb{R}\backslash\{0\}, \tag{2.8}$$

$$\|Q_{\gamma/2}\psi\| \leqslant C \cdot (\|\psi\| + \|\psi\|^{1-\delta}\, \|\nabla\psi\|^\delta), \qquad \textit{for} \quad \psi \in H^{1,0}, \tag{2.9}$$

where $C = C(n, q, \gamma)$ and $\delta = \delta(q)$.

(2) *Let q satisfy $\gamma < \delta(q) < \min(2, n/2)$. Then*

$$\|Q_\gamma\psi\| \leqslant C \cdot (\|\psi\| + \|\psi\|_q), \qquad \textit{for} \quad \psi \in L^2 \cap L^q, \tag{2.10}$$

$$\|Q_\gamma\psi\| \leqslant C \cdot (\|\psi\| + |t|^{-\delta}\, \|\psi\|^{1-\delta/2}\, \|J^2\psi\|^{\delta/2}), \qquad \textit{for} \quad \psi \in H^{2,2} \textit{ and } t \in \mathbb{R}\backslash\{0\}, \tag{2.11}$$

where $C = C(n, q, \gamma)$ and $\delta = \delta(q)$.

Proof. We first note that $Q_\sigma \in L^p + L^\infty$ for $1 \leqslant p < n/\sigma$. Thus (2.7) and (2.10) follow from Hölder's inequality. A simple application of Lemma 2.1 to (2.7) proves (2.8)–(2.9). Similarly, we obtain (2.11) from (2.10) if we note that $\|\partial^\alpha\psi\|$ with $|\alpha| = 2$ is estimated by $C\, \|\Delta\psi\|$. Q.E.D.

LEMMA 2.5. (1) *Let $n \geqslant 3$, $0<\gamma<2$, and $r=2n/(n-\gamma)$. Then for any $m \in \mathbb{N} \cup \{0\}$ there exist $0 \leqslant b_j \leqslant 1$ $(1 \leqslant j \leqslant 3)$ depending only on m, n, γ such that $\sum_{j=1}^{3} b_j = 1$,*

$$\begin{aligned}\sum_{|\alpha|=m} &\|\partial^\alpha((Q_\gamma * (\psi_1\overline{\psi_2}))\psi_3)\| \\ &\leqslant C \prod_{j=1}^{3} \left(\|\psi_j\|_r^{1-b_j} \sum_{|\alpha|=m} \|\partial^\alpha \psi_j\|^{b_j} \right) \\ &\quad + C \prod_{j=1}^{2} (\|\nabla\psi_j\|^{\gamma/2} \|\psi_j\|^{1-\gamma/2}) \cdot \sum_{|\alpha|=m} \|\partial^\alpha \psi_3\|, \\ &\quad \textit{for} \quad \psi_j \in H^{l,m}\ (1 \leqslant j \leqslant 3), \textit{ where } l = \max(m, 1), \end{aligned} \tag{2.12}$$

$$\begin{aligned}\sum_{|\alpha|=m} &\|J^\alpha((Q_\gamma * (\psi_1\overline{\psi_2}))\psi_3)\| \\ &\leqslant C \prod_{j=1}^{3} \left(\|\psi_j\|_r^{1-b_j} \sum_{|\alpha|=m} \|J^\alpha \psi_j\|^{b_j} \right) \\ &\quad + C \prod_{j=1}^{2} (\|\nabla\psi_j\|^{\gamma/2} \|\psi_j\|^{1-\gamma/2}) \cdot \sum_{|\alpha|=m} \|J^\alpha \psi_3\|, \\ &\quad \textit{for} \quad \psi_j \in H^{l,m}\ (1 \leqslant j \leqslant 3) \textit{ and } t \in \mathbb{R}, \end{aligned} \tag{2.13}$$

$$\begin{aligned}\sum_{|\alpha|=m} &\|J^\alpha((Q_\gamma * (\psi_1\overline{\psi_2}))\psi_3)\| \\ &\leqslant C \prod_{j=1}^{3} \left(\|\psi_j\|_r^{1-b_j} \sum_{|\alpha|=m} \|J^\alpha \psi_j\|^{b_j} \right) \\ &\quad + C |t|^{-\gamma} \prod_{j=1}^{2} (\|J\psi_j\|^{\gamma/2} \|\psi_j\|^{1-\gamma/2}) \cdot \sum_{|\alpha|=m} \|J^\alpha \psi_3\|, \\ &\quad \textit{for} \quad \psi_j \in H^{l,l}\ (1 \leqslant j \leqslant 3) \textit{ and } t \in \mathbb{R}\backslash\{0\}. \end{aligned} \tag{2.14}$$

(2) *Let γ, p, q, and r satisfy $0<\gamma<n$, $r=2n/(n-\gamma)$, and $2 \leqslant q<r<p<\infty$. Then for any $m \in \mathbb{N} \cup \{0\}$ there exist $0 \leqslant b_j \leqslant 1$ $(1 \leqslant j \leqslant 3)$ depending only on m, n, γ such that $\sum_{j=1}^{3} b_j = 1$,*

$$\begin{aligned}\sum_{|\alpha|=m} &\|\partial^\alpha((Q_\gamma * (\psi_1\overline{\psi_2}))\psi_3)\| \\ &\leqslant C \prod_{j=1}^{3} \left(\|\psi_j\|_r^{1-b_j} \sum_{|\alpha|=m} \|\partial^\alpha \psi_j\|^{b_j} \right) \\ &\quad + C \left(\prod_{j=1}^{2} \|\psi_j\|_p + \prod_{j=1}^{2} \|\psi_j\|_q \right) \sum_{|\alpha|=m} \|\partial^\alpha \psi_3\|, \\ &\quad \textit{for} \quad \psi_j \in H^{m,0} \cap L^p\ (1 \leqslant j \leqslant 3), \end{aligned} \tag{2.15}$$

$$\sum_{|\alpha|=m} \|J^\alpha((Q_\gamma * (\psi_1\overline{\psi_2}))\psi_3)\|$$

$$\leqslant C\prod_{j=1}^{3}\left(\|\psi_j\|_r^{1-b_j}\sum_{|\alpha|=m}\|J^\alpha\psi_j\|^{b_j}\right)$$

$$+C\left(\prod_{j=1}^{2}\|\psi_j\|_p+\prod_{j=1}^{2}\|\psi_j\|_q\right)\sum_{|\alpha|=m}\|J^\alpha\psi_3\|,$$

$$\textit{for}\quad \psi_j\in H^{m,m}\cap L^p\ (1\leqslant j\leqslant 3)\ \textit{and}\ t\in\mathbb{R}. \tag{2.16}$$

Proof. For part (2), see N. Hayashi and Y. Tsutsumi [9, Lemma 2.7, (2.7)–(2.8)]. We prove part (1). Let $\phi_j = S^{-1}\psi_j$ for $t \neq 0$. Then

$$\sum_{|\alpha|=m} \|J^\alpha((Q_\gamma * (\psi_1\overline{\psi_2}))\psi_3)\|$$

$$=|t|^m\sum_{|\alpha|=m}\|\partial^\alpha((Q_\gamma*(\phi_1\overline{\phi_2}))\phi_3)\|\leqslant I_1+I_2,$$

where

$$I_1=|t|^m\sum_{|\alpha|=m}\|(Q_\gamma*(\phi_1\overline{\phi_2}))\,\partial^\alpha\phi_3\|,$$

$$I_2=C\,|t|^m\sum_{\substack{|\alpha_1+\alpha_2+\alpha_3|=m\\ |\alpha_3|\leqslant m-1}}\|(Q_\gamma*(\partial^{\alpha_1}\phi_1\,\overline{\partial^{\alpha_2}\phi_2}))\,\partial^{\alpha_3}\phi_3\|.$$

I_1 is estimated by

$$|t|^m\,\|Q_\gamma*(\phi_1\overline{\phi_2})\|_\infty\sum_{|\alpha|=m}\|\partial^\alpha\phi_3\|$$

$$=\|Q_\gamma*(\psi_1\overline{\psi_1})\|_\infty\sum_{|\alpha|=m}\|J^\alpha\psi_3\|$$

$$\leqslant\begin{cases} C\,|t|^{-\gamma}\displaystyle\prod_{j=1}^{2}(\|J\psi_j\|^{\gamma/2}\,\|\psi_j\|^{1-\gamma/2})\cdot\sum_{|\alpha|=m}\|J^\alpha\psi_3\|,\\[2ex] C\displaystyle\prod_{j=1}^{2}(\|\nabla\psi_j\|^{\gamma/2}\,\|\psi_j\|^{1-\gamma/2})\cdot\sum_{|\alpha|=m}\|J^\alpha\psi_3\|,\end{cases}$$

since

$$\begin{aligned}
&\left|\int |x-y|^{-\gamma}\,\psi_1\overline{\psi_2}\,dy\right| \\
&\leqslant \prod_{j=1}^{2}\left(\int |x-y|^{-2}\,|\psi_j|^2\,dy\right)^{\gamma/4}\,\|\psi_j\|^{1-\gamma/2} \\
&\leqslant \begin{cases} C\,|t|^{-\gamma}\displaystyle\prod_{j=1}^{2}\|J\psi_j\|^{\gamma/2}\,\|\psi_j\|^{1-\gamma/2}, \\ C\displaystyle\prod_{j=1}^{2}\|\nabla\psi_j\|^{\gamma/2}\,\|\psi_j\|^{1-\gamma/2}, \end{cases}
\end{aligned}$$

where we have used Hölder's inequality and Lemma 2.2. We next estimate I_2. By Hölder's inequality and Lemmas 2.1–2.2, I_2 is estimated by $C\prod_{j=1}^{3}(\|\psi_j\|_r^{1-b_j}\sum_{|\alpha|=m}\|J^\alpha\psi_j\|^{b_j})$, with $0\leqslant b_j\leqslant 1$ depending only on m, n, γ such that $\sum_{j=1}^{3}b_j=1$ (see N. Hayashi and Y. Tsutsumi [9, Lemma 2.7]). Inequalities (2.13)–(2.14) now follow by collecting these estimates, provided $t\neq 0$. The case $t=0$ can be proved similarly. Q.E.D.

LEMMA 2.6. *Let $n\geqslant 3$ and $1\leqslant\gamma\leqslant 2$. Then for $\psi_j\in H^{1,0}$ $(1\leqslant j\leqslant 3)$,*

$$\|Q_\gamma*(\psi_1\psi_1)\|_\infty\leqslant C\,\|\nabla\psi_1\|\,\|\nabla\psi_2\|^{\gamma-1}\,\|\psi_2\|^{2-\gamma}, \tag{2.17}$$

$$\|(Q_\gamma*(\psi_1\psi_2))\psi_3\|\leqslant C\,\|\psi_1\|^{2-\gamma}\,\|\nabla\psi_1\|^{\gamma-1}\prod_{j=2}^{3}\|\nabla\psi_j\|. \tag{2.18}$$

Proof. Inequality (2.17) follows from Hölder's inequality and Lemma 2.2. Indeed,

$$\begin{aligned}
\int |x-y|^{-\gamma}\,|\psi_1\psi_2|\,dy &\leqslant \left(\int |x-y|^{-1}\,|\psi_1|^2\,dy\right)^{1/2} \\
&\quad\times\left(\int |x-y|^{-2}\,|\psi_2|^2\,dy\right)^{(\gamma-1)/2}\,\|\psi_2\|^{2-\gamma} \\
&\leqslant C\,\|\nabla\psi_1\|\,\|\nabla\psi_2\|^{\gamma-1}\,\|\psi_2\|^{2-\gamma}.
\end{aligned}$$

Inequality (2.18) follows from Hölder's inequality and Lemmas 2.1–2.2. Indeed,

$$\begin{aligned}
\|(Q_{\gamma+1}*(\psi_1\psi_2))\psi_3\| &\leqslant \|Q_{\gamma+1}*(\psi_1\psi_2)\|_n\,\|\psi_3\|_{2n/(n-2)} \\
&\leqslant C\,\|\psi_1\|_{2n(n+2-2\gamma)}\prod_{j=2}^{3}\|\psi_j\|_{2n/(n-2)} \\
&\leqslant C\,\|\psi_1\|^{2-\gamma}\,\|\nabla\psi_1\|^{\gamma-1}\prod_{j=2}^{3}\|\nabla\psi_j\|. \quad \text{Q.E.D.}
\end{aligned}$$

Notation 2 (After T. Kato [12] and K. Yajima [15]). For an interval $I \subset \mathbb{R}$ and $0 \leqslant \delta(q) < 1$, $X(I; q)$ denotes the Banach space defined by

$$X(I; q) = \{ v \in C(I; L^2) \cap L^{q, \theta(q)}(I);$$
$$|||v|||_{X(I;q)} = \|v\|_{2,\infty;I} + \|v\|_{q,\theta(q);I} < \infty \}.$$

For a compact interval $I = [-a, a]$ with $a > 0$, we abbreviate $X(I; q)$ and $|||\cdot|||_{X(I;q)}$ by $X(a; q)$ and $|||\cdot|||_{X(a;q)}$, respectively.

The following lemma will be crucial in the sequel.

LEMMA 2.7. *Let $t_0 \in \mathbb{R}$, $a > 0$, and $I = [t_0 - a, t_0 + a]$. Let q and θ satisfy $0 \leqslant \delta(q) < 1$ and $\theta = \theta(q)$. Then*:

(1) *U maps L^2 into $X(\mathbb{R}; q)$ continuously.*

(2) *G_{t_0} maps $L^{2,1}(I) + L^{q', \theta(q)'}$ into $X(I; q)$ continuously.*

Moreover, there exists a constant $C = C(n, q)$ independent of t_0 and a such that

$$|||G_{t_0} v|||_{X(I;q)} \leqslant C\, |||v|||_{2,1;I}, \qquad \textit{for} \quad v \in L^{2,1}(I), \tag{2.19}$$

$$|||G_{t_0} v|||_{X(I;q)} \leqslant C\, |||v|||_{q',\theta';I}, \qquad \textit{for} \quad v \in L^{q',\theta'}(I). \tag{2.20}$$

For Lemma 2.7, see, e.g., K. Yajima [15].

Remark 2.1. The smoothing property of the operators U and G, which is formulated above, has been used to establish fairly general existence and uniqueness theorems by a simple contraction argument (see T. Kato [12] and K. Yajima [15]). For a class of nonlinear Klein–Gordon equations, J. Ginibre and G. Velo [6] found some function spaces and estimates analogous to those in Lemma 2.7 with which such a contraction argument can be applied.

LEMMA 2.8. *Let γ, q, and θ satisfy $0 < \gamma < \min(2, n)$, $\gamma/2 < \delta(q) < \min(1, n/2)$, and $\theta = \theta(q)$. Let $t_0 \in \mathbb{R}$, $0 < a \leqslant 1$, and $I = [t_0 - a, t_0 + a]$. Then*

$$|||G_{t_0}((Q_\gamma * (v_1 v_2)) v_3)|||_{X(I;q)}$$
$$\leqslant \begin{cases} Ca^{1-\delta} \prod_{j=1}^{2} |||v_j|||_{2,\infty;I} \cdot |||v_3|||_{X(I;q)}, & \textit{for} \quad v_1, v_2 \in L^{2,\infty} \textit{ and } v_3 \in X(I;q), \\ Ca^{1-\delta} |||v_1|||_{X(I;q)} \prod_{j=2}^{3} |||v_j|||_{2,\infty;I}, & \textit{for} \quad v_1 \in X(I;q) \textit{ and } v_2, v_3 \in L^{2,\infty}, \\ Ca^{1-\delta} \prod_{j=1}^{3} |||v_j|||_{X(I;q)}, & \textit{for} \quad v_j \in X(I;q)\ (1 \leqslant j \leqslant 3), \end{cases}$$

where $C = C(n, q, \gamma)$ and $\delta = \delta(q)$.

Proof. Let $p=n/\delta(q)$. We decompose Q_γ as $Q_\gamma=Q_\gamma^{(p)}+Q_\gamma^{(\infty)}$ with $Q_\gamma^{(p)}\in L^p$ and $Q_\gamma^{(\infty)}\in L^\infty$. With Q_γ written in this form, we use Lemma 2.7 and Hölder's and Young's inequalities to obtain

$$\begin{aligned}
&|||G_{t_0}((Q_\gamma*(v_1v_2))v_3)|||_{X(I;q)}\\
&\qquad\leqslant C\,|||(Q_\gamma^{(p)}*(v_1v_2))v_3|||_{q',\theta';I}+C\,|||(Q_\gamma^{(\infty)}*(v_1v_2))v_3|||_{2,1;I},\\
&|||(Q_\gamma^{(p)}*(v_1v_2))v_3|||_{q',\theta';I}\\
&\qquad\leqslant\begin{cases}
Ca^{1-\delta}\,\|Q_\gamma^{(p)}\|_p\prod_{j=1}^{2}|||v_j|||_{2,\infty;I}\cdot|||v_3|||_{q,\theta;I},\\
\text{for}\quad v_1,v_2\in L^{2,\infty}(I),\ v_3\in L^{q,\theta}(I),\\
Ca^{1-\delta}\,\|Q_\gamma^{(p)}\|_p\,|||v_1|||_{q,\theta;I}\prod_{j=2}^{3}|||v_j|||_{2,\infty;I},\\
\text{for}\quad v_1\in L^{q,\theta}(I),\ v_2,v_3\in L^{2,\infty}(I),
\end{cases}\\
&|||(Q_\gamma^{(\infty)}*(v_1v_2))v_3|||_{2,1;I}\\
&\qquad\leqslant Ca\,\|Q_\gamma^{(\infty)}\|_\infty\prod_{j=1}^{3}|||v_j|||_{2,\infty;I},\qquad\text{for}\quad v_j\in L^{2,\infty}(I),\ 1\leqslant j\leqslant 3.
\end{aligned}$$

These estimates prove the lemma. Q.E.D.

LEMMA 2.9. *Let γ, q, θ, and m satisfy $0<\gamma<\min(2,n/2)$, $\gamma/2<\delta(q)<\min(1,n/2)$, $\theta=\theta(q)$, and $m>(1+\gamma)\max(1/(2-\gamma),\ 1/(n/2-\gamma))$. Let t_0, a, and I be as in Lemma 2.8. Let $Q_{\gamma,j}$ be defined by $Q_{\gamma,j}(x)=1/(|x|+1/j)^\gamma$ for $j\in\mathbb{N}$. Then for $v\in X(I;q)$,*

$$\sup_{j\in\mathbb{N}}|||G_{t_0}Q_{\gamma,j}v|||_{X(I;q)}\leqslant Ca^{1-\delta}\,|||v|||_{X(I;q)},\tag{2.21}$$

$$\begin{aligned}|||G_{t_0}(Q_{\gamma,j}-Q_{\gamma,k})v|||_{X(I;q)}&\leqslant Ca^{1-\delta}\,|1/j-1/k|^{\gamma/m}\\&\qquad\times|||v|||_{X(I;q)},\qquad j,k\in\mathbb{N},\end{aligned}\tag{2.22}$$

where $C=C(n,q,\gamma)$ and $\delta=\delta(q)$.

Proof. Let $p=n/\delta(q)$. We decompose $Q_{\gamma,j}$ as $Q_{\gamma,j}=V_j^{(p)}+V_j^{(\infty)}$ with $\sup_{j\in\mathbb{N}}(\|V_j^{(p)}\|_p+\|V_j^{(\infty)}\|_\infty)\leqslant C(n,q,\gamma)$. Then

$$\begin{aligned}|||G_{t_0}Q_{\gamma,j}v|||_{X(I;q)}&\leqslant C\,|||V_j^{(p)}v|||_{q',\theta';I}+C\,|||V_j^{(\infty)}v|||_{2,1;I}\\
&\leqslant Ca^{1-\delta}\,\|V_j^{(p)}\|_p\,|||v|||_{q,\theta;I}+Ca\,\|V_j^{(\infty)}\|_\infty\,|||v|||_{2,1;I}\\
&\leqslant Ca^{1-\delta}\,|||v|||_{X(I;q)},\qquad j\in\mathbb{N},\end{aligned}$$

where we have used Lemma 2.7 and Hölder's inequality. Inequality (2.21) follows from the above estimates. We next prove (2.22) by making a reduction. If we can show that for $0<\alpha\leqslant m$ and $j,\ k\in\mathbb{N}$,

$$|Q_{\gamma,j}-Q_{\gamma,k}|\leqslant(\alpha\,|1/j-1/k|)^{\alpha/m} \times(Q_{\alpha(1+1/m),j}+Q_{\alpha(1+1/m),k}), \tag{2.23}$$

then (2.22) will follow from (2.23) with $\alpha=\gamma$ in the same way as in the proof of (2.21). To verify (2.23), we prove

$$|(r+s)^{-\alpha}-(r+t)^{-\alpha}|\leqslant(\alpha\,|s-t|)^{\alpha/m} \times((r+s)^{-\alpha(1+1/m)}+(r+t)^{-\alpha(1+1/m)}), \tag{2.24}$$

for $0<\alpha\leqslant m$ and $r,\ s,\ t\geqslant 0$. Now let $f(\sigma)=(r+s+\sigma(s-t))^{-\alpha}$ for $\sigma\in[0,1]$. Then

$$\begin{aligned}|f(1)-f(0)|&\leqslant\int_0^1|f'(\sigma)|\,d\sigma\leqslant\alpha\,|s-t|\,(f(1)^{1+1/\alpha}+f(0)^{1+1/\alpha})\\&\leqslant\alpha\,|s-t|\,(f(1)^{1+1/m}+f(0)^{1+1/m})^{(1+1/\alpha)(1+1/m)},\end{aligned} \tag{2.25}$$

since $1+1/m\leqslant 1+1/\alpha$. On the other hand,

$$\begin{aligned}|f(1)-f(0)|&\leqslant\max(f(1),f(0))\\&\leqslant(f(1)^{1+1/m}+f(0)^{1+1/m})^{1/(1+1/m)}.\end{aligned} \tag{2.26}$$

From (2.25)–(2.26) we have

$$|f(1)-f(0)|\leqslant(\alpha\,|s-t|)^{\alpha/m}\,(f(1)^{1+1/m}+f(0)^{1+1/m}),$$

which is precisely (2.24). Q.E.D.

LEMMA 2.10. *Let $n\geqslant 2$. Let γ, q, and θ satisfy $0<\gamma<\min(2,n/2)$, $\gamma/2<\delta(q)<\min(1,n/2)$, and $\theta=\theta(q)$. Let m, t_0, a, I, and $Q_{\gamma,j}$ be as in Lemma 2.9. Then for $v\in X(I;q)$ with $Jv\in X(I;q)$,*

$$\sup_{j\in\mathbb{N}}|||G_{t_0}JQ_{\gamma,j}v|||_{X(I;q)} \leqslant Ca^{1-\delta}(|||Jv|||_{X(I;q)}+(1+|t_0|)\,|||v|||_{X(I;q)}), \tag{2.27}$$

$$|||G_{t_0}J(Q_{\gamma,j}-Q_{\gamma,k})v|||_{X(I;q)} \leqslant Ca^{1-\delta}\,|1/j-1/k|^{\gamma/m} \times(|||Jv|||_{X(I;q)}+(1+|t_0|)\,|||v|||_{X(I;q)}),\qquad j,k\in\mathbb{N}, \tag{2.28}$$

Proof. Since $Ju \in X(I; q)$, (2.21) leads to

$$|||G_{t_0} Q_{\gamma,j} Jv|||_{X(I;q)} \leqslant Ca^{1-\delta} \, |||Jv|||_{X(I;q)}.$$

Therefore (2.27) will bollow if we can show that

$$\begin{aligned} &|||G_{t_0}[J, Q_{\gamma,j}]\, v|||_{X(I;q)} \\ &\qquad \leqslant Ca^{1-\delta}(|||Jv|||_{X(I;q)} + (1+|t_0|)\, |||v|||_{X(I;q)}), \end{aligned} \tag{2.29}$$

where $[J, Q_{\gamma,j}] = JQ_{\gamma,j} - Q_{\gamma,j}J = -it\gamma\hat{x}Q_{\gamma+1,j}$. To prove (2.29) we distinguish between two cases: (1) $n \geqslant 4$. (2) $n \leqslant 3$.

(1) When $n \geqslant 4$, we decompose $Q_{\gamma,j}$ as $Q_{\gamma,j} = V_j^{(r)} + V_j^{(\infty)}$ with $\sup_{n\in\mathbb{N}} (\|V_j^{(r)}\|_r + \|V_j^{(\infty)}\|_\infty) \leqslant C(n, q, \gamma)$, where $r = n/(2\delta(q)+1)$. Note that $1/q' = 1/r + 1/q - 1/n$, $1/q - 1/n = -\delta(q)/n + \frac{1}{2} - 1/n > \frac{1}{2} - 2/n \geqslant 0$. Then we use Hölder's inequality and Lemma 2.1 to obtain

$$\|tV_j^{(r)}v\|_{q'} \leqslant C \, \|V_j^{(r)}\|_r \, \|Jv\|_q,$$

so that the L.H.S. of (2.29) is dominated by

$$\begin{aligned} &C(a^{1-\delta} \, \|V_j^{(r)}\|_r \, |||Jv|||_{q,\theta;I} + a(1+|t_0|) \, \|V_j^{(\infty)}\|_\infty \, |||v|||_{2,\infty,I}) \\ &\qquad \leqslant Ca^{1-\delta}(|||Jv|||_{X(I;q)} + (1+|t_0|)\, |||v|||_{X(I;q)}), \end{aligned}$$

as required.

(2) When $n = 2$ or 3, we decompose $Q_{\gamma,j}$ as $Q_{\gamma,j} = V_j^{(r)} + V_j^{(\infty)}$ with $\sup_{j\in\mathbb{N}} (\|V_j^{(r)}\|_r + \|V_j^{(\infty)}\|_\infty) \leqslant C(n, q, \gamma)$, where $r = n/(2\delta(q) + n/2 - 1)$. Note that $r \geqslant 1$, $1/q' = 1/r + (1-\delta)/n$. Then we use Hölder's inequality and Lemma 2.1 to obtain

$$\|tV_j^{(r)}v\|_{q'} \leqslant \|V_j^{(r)}\|_r \, \|tv\|_{n/(1-\delta)},$$

$$\|tv\|_{n/(1-\delta)} \leqslant \begin{cases} C\, \|Jv\|_q^{1/2} \, \|tv\|_q^{1/2} \leqslant C(\|Jv\|_q + \|tv\|_q), & \text{if} \quad n=3; \\ C\, \|Jv\|^{\delta} \, \|tv\|^{1-\delta} \leqslant C(\|Jv\| + \|tv\|), & \text{if} \quad n=2, \end{cases}$$

so that (2.29) follows from these estimates.

We now turn to (2.28). Inequality (2.22) implies

$$\begin{aligned} &|||G_{t_0}(Q_{\gamma,j} - Q_{\gamma,k})\, Jv|||_{X(I;q)} \\ &\qquad \leqslant Ca^{1-\delta} \, |1/j - 1/k|^{\gamma/m} \, |||Jv|||_{X(I;q)}. \end{aligned}$$

We deduce from (2.23) with $\alpha = \gamma + 1$ that

$$
\begin{aligned}
|[J, Q_{\gamma,j}-Q_{\gamma,k}]| &\leqslant \gamma\,|t|\,|Q_{\gamma+1,j}-Q_{\gamma+1,k}| \\
&\leqslant \gamma(\gamma+1)\,|1/j-1/k|^{(\gamma+1)/m} \\
&\quad\times (Q_{(\gamma+1)(1+1/m),j}+Q_{(\gamma+1)(1+1/m),k}) \\
&\leqslant \gamma(\gamma+1)\,|1/j-1/k|^{\gamma/m} \\
&\quad\times (Q_{(\gamma+1)(1+1/m),j}+Q_{(\gamma+1)(1+1/m),k}).
\end{aligned}
$$

The preceding argument therefore gives (2.28). Q.E.D.

LEMMA 2.11. *Let* $v \in C(\mathbb{R}\backslash\{0\}; L^2)$. *Then*:

(1) $S^{-1}v \in C(\mathbb{R}\backslash\{0\}; L^2)$.

(2) *Let* $k \in \mathbb{N}$. *Assume in addition that* $J^\alpha v \in C(\mathbb{R}\backslash\{0\}; L^2)$ *for* $|\alpha| \leqslant k$. *Then* $S^{-1}v$, $v \in C(\mathbb{R}\backslash\{0\}; L^p)$ *for all* p *such that* $0 \leqslant \delta(p) \leqslant k$ $(n \geqslant 2k+1)$; $0 \leqslant \delta(p) < k$ $(n = 2k)$; $0 \leqslant \delta(p) \leqslant n/2$ $(n \leqslant 2k-1)$.

Proof. (1) The result follows from the inequality

$$
\begin{aligned}
&\|S(-t)\,v(t)-S(-\tau)\,v(\tau)\| \\
&\quad\leqslant \|v(t)-v(\tau)\| + \|(S(-t)-S(-\tau))\,v(\tau)\|. \qquad (2.30)
\end{aligned}
$$

(2) We use induction on k. Let $k=1$. By Lemma 2.1 we obtain

$$\|v(t)\|_p = \|S(-t)\,v(t)\|_p \leqslant C\,\|v(t)\|^{1-\delta} \left\| \frac{1}{t} Jv(t) \right\|^{\delta}, \qquad (2.31)$$

$$
\begin{aligned}
\|S(-t)\,v(t)-S(-\tau)\,v(\tau)\|_p &\leqslant C\,\|S(-t)\,v(t)-S(-\tau)\,v(\tau)\|^{1-\delta} \\
&\quad\times \left\| S(-t)\frac{1}{t}Jv(t) - S(-\tau)\frac{1}{\tau}Jv(\tau) \right\|^{\delta}, \qquad (2.32)
\end{aligned}
$$

where $\delta = \delta(p)$. From (2.30), (2.31), and part (1), it follows that $S^{-1}v \in C(\mathbb{R}\backslash\{0\}; L^p)$. An inequality similar to (2.30) implies that $v \in C(\mathbb{R}\backslash\{0\}; L^p)$. This proves part (2) for $k=1$. Let $k \geqslant 2$ and assume that part (2) holds for $k-1$. Let v satisfy $J^\alpha v \in C(\mathbb{R}\backslash\{0\}; L^p)$ for $|\alpha| \leqslant k$. The result then follows from the induction hypothesis and the following inequalities derived from Lemma 2.1:

$$\|\psi\|_\infty \leqslant C \sum_{|\alpha|=k} \|\partial^\alpha \psi\|^{n/(q+n)}\,\|\psi\|_q^{q/(q+n)},$$

$$\text{if} \quad n = 2k-2,\ 0 \leqslant \delta(q) < k-1;$$

$$\|\psi\|_p \leqslant C \sum_{|\alpha|=k} \|\partial^\alpha \psi\|^a\,\|\psi\|_{2n}^{1-a},$$

$$\text{if} \quad n = 2k-1,\ \delta(p) = n/2 - \tfrac{1}{2} + a,\ 0 < a \leqslant \tfrac{1}{2};$$

$$\|\psi\|_p \leqslant C \sum_{|\alpha|=k} \|\partial^\alpha \psi\|^a \|\psi\|_n^{1-a},$$

$$\text{if} \quad n=2k,\ \delta(p)=n/2-1+a,\ 0<a<1;$$

$$\|\psi\|_p \leqslant C \sum_{|\alpha|=k} \|\partial^\alpha \psi\|^a \|\psi\|_q^{1-a},$$

$$\text{if} \quad n \geqslant 2k+1,\ \delta(q)=k-1,\ \delta(p)=k-1+a,\ 0<a\leqslant 1.$$

Q.E.D.

3. Proofs of Theorems 1–5

In order to derive various identities for solutions of (**), we need a regularization procedure. We first study the existence of sufficiently smooth solutions of regularized equations for (**).

PROPOSITION 3.1. *Let* $V \in \bigcap_{m \in \mathbb{N}} H_\infty^{m,0}$ *be a real function. Let* $\phi \in \mathscr{S}$. *Then there exists a unique solution* $u \in C^1(\mathbb{R}; \mathscr{S})$ *satisfying*

$$i\partial_t u + \tfrac{1}{2}\Delta u = Vu + (V_2 * |u|^2)u \equiv \tilde{F}(u) \qquad in \quad C(\mathbb{R}; \mathscr{S}), \tag{3.1}$$

$$u(0)=\phi. \tag{3.2}$$

Moreover, u satisfies the identities

$$\|u(t)\| = \|\phi\|, \qquad t \in \mathbb{R}, \tag{3.3}$$

$$E(t)=E(0), \qquad t \in \mathbb{R}, \tag{3.4}$$

where

$$E(t) = \|\nabla u(t)\|^2 + 2(Vu(t), u(t)) + ((V_2 * |u|^2)\, u(t), u(t)).$$

Remark 3.1. There are related results by W. Hunziker [10], N. Hayashi, K. Nakamitsu, and M. Tsutsumi [7], and N. Hayashi and Y. Tsutsumi [9].

Proof of Proposition 3.1. For integers $m \geqslant 2$, $s \geqslant 0$, and a compact interval $I=[-a, a]$, $a>0$, we define the following Banach space $B^{m,s}(I)$ by $B^{m,s}(I)=\{v \in C(I; L^2);\ \ U^{-1}v \in C(I; H^{m,0} \cap H^{0,s})\}$ with the norm $|||v|||_{B^{m,s}(I)} = \sup_{t \in I} (\|v(t)\|_{m,0} + \|U(-t)\, v(t)\|_{0,s})$. Since $x^\alpha U(-t) = U(-t) J^\alpha$, $|||\cdot|||_{B^{m,s}(I)}$ is equivalent to the norm $|||v||| = \sup_{t \in I} (\sum_{|\alpha| \leqslant m} \|\partial^\alpha v(t)\| + \sum_{|\beta| \leqslant s} \|J^\beta v(t)\|)$. For $v \in B^{m,s}(I)$ we define

$$Mv = U\phi - iG\tilde{F}(v). \tag{3.5}$$

We first prove that $U^{-1}\tilde{F}(v) \in L^\infty(I;\ H^{m,0} \cap H^{0,s})$. We estimate $U(-t)\,\tilde{F}(v(t))$ in $H^{m,0}$ by

$$\begin{aligned}\|U(-t)\,\tilde{F}(v(t))\|_{m,0} &\leqslant C \sum_{|\alpha| \leqslant m} \|\partial^\alpha \tilde{F}(v(t))\| \\ &\leqslant C(\|V\|_{m,0,\infty} + \|v(t)\|_{1,0}^2)\,\|v(t)\|_{m,0},\end{aligned} \tag{3.6}$$

where we have used (2.6), (2.15) with p and q satisfying

$$\max(0, \tfrac{1}{2} - 1/n) < 1/p < \tfrac{1}{2} - \max(\gamma_2, \gamma_3)/2n < 1/q \leqslant \tfrac{1}{2}, \tag{3.7}$$

and Lemma 2.1. We estimate $U(-t)\,\tilde{F}(v(t))$ in $H^{0,s}$ by

$$\begin{aligned}\|U(-t)\,\tilde{F}(v(t))\|_{0,s} &\leqslant C \sum_{|\beta| \leqslant s} \|J^\beta \tilde{F}(v(t))\| \\ &\leqslant C(\|V\|_{s,0,\infty}(1 + |t|^s) \\ &\qquad + \|v(t)\|_{1,0}^2)\,\|U(-t)\,v(t)\|_{0,s},\end{aligned} \tag{3.8}$$

where we have used (2.6), (2.15) with p and q satisfying (3.7), and Lemma 2.1. Thus (3.6) and (3.8) prove our claim. Moreover, we have proved that $Mv \in B^{m,s}(I)$,

$$U^{-1}\,\partial^\alpha Mv = \partial^\alpha \phi - iU^{-1}G\,\partial^\alpha \tilde{F}(v), \qquad |\alpha| \leqslant m, \tag{3.9}$$

$$U^{-1}J^\alpha Mv = x^\alpha \phi - iU^{-1}GJ^\beta \tilde{F}(v), \qquad |\beta| \leqslant s. \tag{3.10}$$

We denote by B_ρ the closed ball in $B^{m,s}(I)$ with radius $\rho > 0$. Let $\rho = 2(\|\phi\|_{m,0} + \|\phi\|_{0,s})$. We show that if a is small enough, then the map $v \mapsto Mv$ preserves B_ρ and is a contraction on B_ρ. Let $v \in B_\rho$. Then by (3.5), (3.6), and (3.8) we obtain

$$\begin{aligned}|||Mv|||_{B^{m,s}(I)} &\leqslant \|\phi\|_{m,0} + \|\phi\|_{0,s} + a\,|||\tilde{F}(v)|||_{B^{m,s}(I)} \\ &\leqslant \rho/2 + aC(1 + a^s + |||v|||_{B^{1,0}(I)}^2)\,|||v|||_{B^{m,s}(I)} \\ &\leqslant \rho/2 + aC(1 + a^s + \rho^2)\rho.\end{aligned} \tag{3.11}$$

Let $v_1, v_2 \in B_\rho$ and let $w = v_1 - v_2$. By (3.9) and (3.10) we write

$$\begin{aligned}\partial^\alpha(Mv_1 - Mv_2) &= -iG\,\partial^\alpha(\tilde{F}(v_1) - \tilde{F}(v_2)) \\ &= -iG\,\partial^\alpha(Vw) - iG\,\partial^\alpha((V_2 * |v_1|^2)w \\ &\qquad + (V_2 * (\bar{w}v_1))v_2 + (V_2 * (\overline{v_2}w))v_2), \\ J^\alpha(Mv_1 - Mv_2) &= -iGJ^\beta(\tilde{F}(v_1) - \tilde{F}(v_2)) \\ &= -iGJ^\beta(Vw) - iGJ^\beta((V_2 * |v_1|^2)w \\ &\qquad + (V_2 * (\bar{w}v_1))v_2 + (V_2 * (\overline{v_2}w))v_2),\end{aligned}$$

so that we have

$$|||Mv_1 - Mv_2|||_{B^{m,s}(I)} \leqslant aC(1 + a^s + \rho^2) \, |||v_1 - v_2|||_{B^{m,s}(I)} \tag{3.12}$$

in the same way as before. Inequalities (3.11) and (3.12) prove that for some a_0 small enough, M is a contraction on B_ρ, and hence M has a unique fixed point $u \in B_\rho$. Let $I_0 = [-a_0, a_0]$. That unique fixed point $u \in B^{m,s}(I_0)$ satisfies the integral equation

$$u = U\phi - iG\tilde{F}(u) \tag{3.13}$$

in $B^{m,s}(I_0)$. It follows from (3.6) and (3.13) that u satisfies the differential equation

$$i\partial_t u + \tfrac{1}{2} \Delta u = \tilde{F}(u) \tag{3.14}$$

in $C(I_0; H^{m-2,0})$. This implies the conservation laws

$$\|u(t)\| = \|\phi\|, \tag{3.15}$$

$$E(t) = E(0), \tag{3.16}$$

for $t \in I_0$. These identities give an a priori estimate of the solution u in $H^{1,0}$. Indeed, by the definition of $E(t)$ and by Lemmas 2.1–2.2,

$$\begin{aligned} \|\nabla u(t)\|^2 \leqslant{} & E(t) + 2 \, \|V\|_\infty \, \|u(t)\|^2 \\ & + C \sum_{k=2}^{3} \|u(t)\|^{4-\gamma_k} \, \|\nabla u(t)\|^{\gamma_k}, \end{aligned} \tag{3.17}$$

which when combined with (3.16)–(3.17) leads to the desired estimates

$$\|u(t)\|_{1,0} \leqslant C(\|\phi\|_{1,0}), \tag{3.18}$$

for $t_0 \in I_0$, since $\gamma_2, \gamma_3 < 2$. In view of (3.6), (3.8), (3.11)–(3.18), and the relation

$$U(-t)\, u(t) = U(-\tau)\, u(\tau) - i \int_\tau^t U(-s)\, \tilde{F}(u(s))\, ds, \tag{3.19}$$

the existence of global solution in $B^{m,s} = \{v \in C(\mathbb{R}; H^{m,0});\ U^{-1}v \in C(\mathbb{R}; H^{0,s})\}$ will be established if we can derive a priori control in $B^{m,s}$ of the solution from a priori estimates of the $H^{1,0}$-norm. For this purpose we prove that every solution $u \in B^{m,s}([-a, a])$ of (3.13) with $a > 0$ satisfies the estimate

$$|||u|||_{B^{m,s}([-a,a])} \leqslant C \cdot (\|\phi\|_{m,0} + \|\phi\|_{0,s}) \cdot \exp(\dot{C}(\|\phi\|_{1,0}) \cdot a). \tag{3.20}$$

Let $a>0$ and let $u\in B^{m,s}([-a,a])$ satisfy (3.13) for $|t|\leqslant a$. Then u satisfies (3.15), (3.16), and (3.18) in the same t-interval, so that (3.20) follows from (3.6), (3.8), (3.19) with τ replaced by 0, and Gronwall's inequality.

We have thus proved that there exists $u\in C_b(\mathbb{R}; H^{1,0})\cap B^{m,s}$ satisfying (3.19) in $H^{m,0}\cap H^{0,s}$. Since $\mathscr{S}=\bigcap_{m,s\in\mathbb{N}}(H^{m,0}\cap H^{0,s})$, we have $U^{-1}u\in C(\mathbb{R};\mathscr{S})$. Moreover, from the expression $u(t)=\mathscr{F}\exp(-i(t/2)\,|\xi|^2)\,\mathscr{F}^{-1}U(-t)\,u(t)$, we find $u\in C(\mathbb{R};\mathscr{S})$. Then it follows from the integral equation that $u\in C^1(\mathbb{R};\mathscr{S})$ and that u satisfies the differential equation (3.1). The uniqueness of solutions follows by a standard argument (see also the proof of Theorem 1). Q.E.D.

Proposition 3.1 gives the existence and uniqueness for solutions of the following regularized equation with $\phi_j\in\mathscr{S}$, $j\in\mathbb{N}$,

$$\begin{aligned} &i\partial_t u_j+\tfrac{1}{2}\Delta u_j=F_j(u_j) \quad \text{in} \quad C(\mathbb{R};\mathscr{S}),\\ &u_j(0)=\phi_j, \end{aligned} \tag{\#}$$

where the regularized interaction F_j is written as

$$F_j(u_j)=V_{1,j}u_j+(V_2*|u_j|^2)u_j \qquad \text{with} \quad V_{1,j}=\lambda_1/(|x|+1/j)^{\gamma_1}.$$

We already know that $u_j\in C^1(\mathbb{R};\mathscr{S})$ and u_j satisfies

$$u_j=U\phi_j-iGF_j(u_j) \quad \text{in} \quad C(\mathbb{R};\mathscr{S}), \tag{3.21}$$

$$\|u_j(t)\|=\|\phi_j\|, \qquad t\in\mathbb{R}, \tag{3.22}$$

$$E_j(t)=E_j(0), \qquad t\in\mathbb{R}, \tag{3.23}$$

where

$$E_j(t)=\|\nabla u_j(t)\|^2+2(V_{1,j}u_j(t),u_j(t))+((V_2*|u_j|^2)\,u_j(t),u_j(t)).$$

In the sequel we do not indicate the dependence of various constants on n, $\{\lambda_k\}$, $\{\gamma_k\}$, and we omit the time variable t of $u(t)$, $u_j(t)$, $v(t)$, $v_j(t)$, etc., when this will not cause confusion. With each $q\in\Gamma$ we associate θ and η by $\theta=\theta(q)$ and $\eta=1-\delta(q)$, respectively. In what follows we define $m_0=(1+\gamma_1)\cdot\max(1/(2-\gamma_1),\,1/(n/2-\gamma_1))$.

In order to study the boundedness and convergence of approximate solutions, it is convenient to work in $X(I;q)$ and its subspaces defined by the following

Notation 3. For $l\in\mathbb{N}$, $q\in\Gamma$, and an interval $I\subset\mathbb{R}$,

$$X_l(I;q)=\{v\in X(I;q);\, J^\alpha v\in X(I;q),\ |\alpha|\leqslant l\},$$

$$Y(I;q)=\{v\in X(I;q);\, J^\alpha v\in C_b(I;L^2),\ |\alpha|\leqslant 2\}.$$

The associated norms are defined respectively by

$$|||v|||_{X_l(I;q)} = \sum_{|\alpha| \leqslant l} |||J^\alpha v|||_{X(I;q)},$$

$$|||v|||_{Y(I;q)} = |||v|||_{X(I;q)} + \sum_{|\alpha| \leqslant 2} |||J^\alpha v|||_{2,\infty;I}.$$

For a compact interval $I = [-a, a]$, $a > 0$, we follow the notational conventions made in Notation 2.

LEMMA 3.1. *Let $\phi \in L^2$ and let $\{\phi_j\}$ be a sequence in $\mathscr{S}$ such that $\phi_j \to \phi$ in L^2 as $j \to \infty$. Let $u_j \in C^1(\mathbb{R}; \mathscr{S})$ be the solution of $(\#)$. Let $q \in \Gamma$. Then for any $T > 0$, there exists a constant $C_T = C(T, q, \sup_{j \in \mathbb{N}} \|\phi_j\|)$ such that*

$$\sup_{j \in \mathbb{N}} |||u_j|||_{X(T;q)} \leqslant C_T, \tag{3.24}$$

$$|||u_j - u_k|||_{X(T;q)} \leqslant C_T(\|\phi_j - \phi_k\| + |1/j - 1/k|^{\gamma_1/m}),\ j, k \in \mathbb{N},\ m > m_0. \tag{3.25}$$

Proof. Let t_0, a, and I be as in Lemma 2.8. Since u_j satisfies

$$u_j(t) = U(t)\, U(-t_0)\, u_j(t_0) - i(G_{t_0} F_j(u_j))(t), \qquad t \in \mathbb{R}, \tag{3.26}$$

we obtain by Lemmas 2.7–2.9 and (3.22),

$$|||u_j|||_{X(I;q)} \leqslant C_1\, \|\phi_j\| + C_2 a^\eta (1 + \|\phi_j\|^2)\, |||u_j|||_{X(I;q)}, \tag{3.27}$$

where $C_1 = C(q)$. We define $C_2 = \sup_{j \in \mathbb{N}} \|\phi_j\|$, $a_1 = \min(1, (2C_1(1 + C_2^2))^{-1/\eta})$. Then (3.27) leads to

$$|||u_j|||_{q,\theta;I} \leqslant |||u_j|||_{X(I;q)} \leqslant 2C_1 C_2, \qquad \text{for} \quad a \in (0, a_1],$$

from which we get

$$|||u_j|||_{q,\theta;[-T,T]} \leqslant 2C_1 C_2 (T/a_1 + 2)^{1/\theta}, \qquad \text{for} \quad T > 0.$$

Thus

$$|||u_j|||_{X(T;q)} \leqslant C_1 C_2 + 2C_1 C_2 (T/a_1 + 2)^{1/\theta}, \qquad \text{for} \quad T > 0. \tag{3.28}$$

This proves (3.24). Now let $v_{jk} = u_j - u_k$. Using (3.26), we write

$$\begin{aligned} v_{jk} = {} & U(\cdot)\, U(-t_0)\, v_{jk}(t_0) - iG_{t_0}(V_{1,k} v_{jk} + (V_{1,j} - V_{1,k}) u_k) \\ & - iG_{t_0}((V_2 * |u_k|^2) v_{jk} \\ & + (V_2 * (v_{jk} \overline{u_k})) u_j + (V_2 * (\overline{v_{jk}} u_k)) u_j). \end{aligned} \tag{3.29}$$

By Lemmas 2.7–2.9 and (3.25) we obtain

$$\begin{aligned}|||v_{jk}|||_{X(I;q)} \leqslant C_3\, &\|v_{jk}(t_0)\| \\ &+ C_3 a^\eta\, |1/j - 1/k|^{\gamma_1/m}\, |||u_k|||_{X(I;q)} \\ &+ C_3 a^\eta (1 + \|\phi_j\|^2 + \|\phi_k\|^2)\, |||v_{jk}|||_{X(I;q)},\end{aligned} \tag{3.30}$$

where $C_3 = C(q)$. We define $a_2 = \min(1, (2C_3(1+2C_2^2))^{-1/\eta})$. Then (3.30) leads to

$$|||v_{jk}|||_{X(I;q)} \leqslant 2C_3\, \|v_{jk}(t_0)\| + |1/j - 1/k|^{\gamma_1/m}\, |||u_k|||_{X(I;q)}, \tag{3.31}$$

for $a \in (0, a_2]$. Let $t_0 = 0$. Then (3.31) gives

$$|||v_{jk}|||_{X(a_2;q)} \leqslant 2C_3\, \|\phi_j - \phi_k\| + |1/j - 1/k|^{\gamma_1/m}\, |||u_k|||_{X(a_2;q)}. \tag{3.32}$$

Thus $\|v_{jk}(\pm a_2)\|$ is estimated in terms of $\|\phi_j - \phi_k\|$ and $|1/j - 1/k|^{\gamma_1/m}$. Moreover, $|||v_{jk}|||_{X([0,2a_2];q)}$ and $|||v_{jk}|||_{X([-2a_2,0];q)}$ can be controlled similarly, if we let $t_0 = a_2$ and $t_0 = -a_2$ in (3.31), respectively. Since a_2 depends only on n, q, $\{\lambda_k\}$, $\{\gamma_k\}$, and $\sup_{j\in\mathbb{N}} \|\phi_j\|$, we iterate this process to obtain estimates for $|||v_{jk}|||_{X([(\pm l-1)a_2,(\pm l+1)a_2];q)}$, $l \in \mathbb{N}$, through $\|\phi_j - \phi_k\|$ and $|1/j - 1/k|^{\gamma_1/m}$. In fact, by induction on $l \in \mathbb{N}$, we see that

$$\begin{aligned}&|||v_{jk}|||_{X([(\pm l-1)a_2,(\pm l+1)a_2];q)} \\ &\quad\leqslant C(q, l)\, \|\phi_j - \phi_k\| + C(\gamma, l) \\ &\qquad\times |1/j - 1/k|^{\gamma_1/m}\, |||u_k|||_{X((l+1)a_2;q)}.\end{aligned} \tag{3.33}$$

Inequalities (3.24) and (3.33) then yield (3.25). Q.E.D.

LEMMA 3.2. *Let $n \geqslant 2$. Let $\phi \in H^{0,1}$ and let $\{\phi_j\}$ be a sequence in $\mathscr{S}$ such that $\phi_j \to \phi$ in $H^{0,1}$ as $j \to \infty$. Let $u_j \in C^1(\mathbb{R}; \mathscr{S})$ be the solution of (#). Let $q \in \Gamma$. Then for any $T > 0$, there exists a constant $C_T = C(T, q, \sup_{j\in\mathbb{N}} \|\phi_j\|_{0,1})$ such that*

$$\sup_{j\in\mathbb{N}} |||u_j|||_{X_1(T;q)} \leqslant C_T, \tag{3.34}$$

$$|||u_j - u_k|||_{X_1(T;q)} \leqslant C_T(\|\phi_j - \phi_k\|_{0,1} + |1/j - 1/k|^{\gamma_1/m}), \quad \textit{for} \quad j, k \in \mathbb{N}, m > m_0. \tag{3.35}$$

Proof. It suffices to prove

$$\sup_{j\in\mathbb{N}} |||Ju_j|||_{X(T;q)} \leqslant C_T, \tag{3.36}$$

$$|||Ju_j - Ju_k|||_{X(T;q)} \leqslant C_T(\|\phi_j - \phi_k\|_{0,1} + |1/j - 1/k|^{\gamma_1/m}). \tag{3.37}$$

Let t_0, a, and I be as in Lemma 2.8. From the relation $[J, V_2 * |u_j|^2] = 2iV_2 * (\mathrm{Im}(\overline{u_j} Ju_j))$, it follows that

$$
\begin{aligned}
(Ju_j)(t) &= U(t)\, U(-t_0)(Ju_j)(t_0) - i(G_{t_0} JF_j(u_j))(t) \\
&= U(t)\, U(-t_0)(Ju_j)(t_0) - i(G_{t_0} JV_{1,j} u_j)(t) \\
&\quad - i(G_{t_0}((V_2 * |u_j|^2)\, Ju_j \\
&\quad - 2(V_2 * (\mathrm{Im}(\overline{u_j} Ju_j)))\, u_j))(t).
\end{aligned} \tag{3.38}
$$

By Lemmas 2.7, 2.8, 2.10, and (3.22), we obtain from (3.38)

$$
\begin{aligned}
|||Ju_j|||_{X(I;q)} \leqslant C_5\, \|Ju_j(t_0)\| \\
+ C_5 a^\eta (1 + \|\phi_j\|^2)\, |||Ju_j|||_{X(I;q)},
\end{aligned} \tag{3.39}
$$

where $C_5 = C(q)$. We define $a_3 = \min(1, (2C_5(1 + C_2^2))^{-1/\eta})$. Let $t_0 = 0$. Then by (3.39), $|||Ju_j|||_{X(a_3;q)}$ is bounded by $2C_5\, \|x\phi_j\|$. Starting from this fact, we obtain (3.34) after finite steps by the same method as in the proof of Lemma 3.1. We now turn to (3.35). Let $v_{jk} = u_j - u_k$. Using (3.38), we write

$$
\begin{aligned}
Jv_{jk} &= U(\cdot)\, U(-t_0)(Jv_{jk})(t_0) \\
&\quad - iG_{t_0} J(V_{1,j} v_{jk} + (V_{1,j} - V_{1,k}) u_k) \\
&\quad - iG_{t_0}((V_2 * |u_j|^2)\, Jv_{jk} \\
&\quad + (V_2 * (\overline{u_j}\, v_{jk} + \overline{v_{jk}} u_k))\, Ju_k) \\
&\quad + 2G_{t_0}((V_2 * (\mathrm{Im}(\overline{v_{jk}}\, Ju_j))) u_j + (V_2 * (\mathrm{Im}(\overline{u_k} Jv_{jk}))) u_j \\
&\quad + (V_2 * (\mathrm{Im}(\overline{u_k} Ju_k))) v_{jk}).
\end{aligned} \tag{3.40}
$$

By Lemmas 2.7, 2.8, 2.10, and (3.22), we obtain from (3.40)

$$
\begin{aligned}
|||Jv_{jk}|||_{X(I;q)} &\leqslant C_6\, \|Jv_{jk}(t_0)\| \\
&\quad + C_6 a^\eta (1 + \|\phi_j\|^2 + \|\phi_k\|^2)\, |||Jv_{jk}|||_{X(I;q)} \\
&\quad + C_6 a^\eta (1 + |t_0|) \\
&\quad \times \left(1 + \sum_{l = k, j} (\|\phi_l\|^2 + |||Ju_l|||^2_{X(I;q)})\right) |||v_{jk}|||_{X(I;q)} \\
&\quad + C_6 a^\eta\, |1/j - 1/k|^{\gamma_1/m}\, (1 + |t_0|)\, |||u_k|||_{X(I;q)},
\end{aligned} \tag{3.41}
$$

where $C_6 = C(q)$. $|||Jv_{jk}|||_{X(I;q)}$ which appears in the second term on the R.H.S. of (3.41) can be handled as before, while the last two terms can be estimated by (3.24), (3.25), and (3.36). Therefore the same argument as in the proof of Lemma 3.1 shows (3.35). Q.E.D.

LEMMA 3.3. *Let $\phi \in H^{0,2}$ and let $\{\phi_j\}$ be a sequence in $\mathscr{S}$ such that $\phi_j \to \phi$ in $H^{0,2}$ as $j \to \infty$. Let $u_j \in C^1(\mathbb{R}; \mathscr{S})$ be the solution of $(\#)$. Then for any $T > 0$, there exists a constant $C_T = C(T, q, \sup_{j \in \mathbb{N}} \|\phi_j\|_{0,2})$ such that*

$$\sup_{j \in \mathbb{N}} |||u_j|||_{Y(I;q)}, \sup_{j \in \mathbb{N}} |||Ku_j|||_{X(I;q)} \leqslant C_T, \tag{3.42}$$

$$\begin{aligned} &|||u_j - u_k|||_{Y(I;q)}, |||Ku_j - Ku_k|||_{X(I;q)} \\ &\quad \leqslant C_T(\|\phi_j - \phi_k\|_{0,2} + |1/j - 1/k|^{\gamma_1/m}), \\ &\quad \textit{for} \quad j, k \in \mathbb{N}, m > m_0. \end{aligned} \tag{3.43}$$

Proof. We first derive an integral equation for Ku_j. We compute (with $v \in C^1(\mathbb{R}; \mathscr{S})$)

$$\begin{aligned} [K, L] &= (J^2 + 2t^2 L)L - L(J^2 + 2t^2 L) \\ &= 2(t^2 L^2 - Lt^2 L) = -4itL, \\ [K, V_{1,j}] &= -2t[A, V_{1,j}] = 2itx \cdot \nabla V_{1,j}, \\ [A, V_2 * |v|^2] &= i(\mathrm{Im}(A(V_2 * |v|^2))) + i(n/2) V_2 * |v|^2 \\ &= i\,\mathrm{Im}(V_2 * (A\,|v|^2) \\ &\qquad + [A, V_2 *]\,|v|^2) + i(n/2)\, V_2 * |v|^2 \\ &= 2iV_2 * \mathrm{Im}(\bar{v}Av) \\ &\qquad + i\,\mathrm{Im}([A, V_2 *]\,|v|^2) + niV_2 * |v|^2 \\ &= 2iV_2 * (\mathrm{Im}(\bar{v}Av)) + [A, V] * |v|^2, \\ [K, V_2 * |v|^2] &= -2t[A, V_2 * |v|^2] + 2it^2\, \partial_t (V_2 * |v|^2) \\ &= -4itV_2 * (\mathrm{Im}(\bar{v}Av)) \\ &\qquad + 4itV_2 * (\mathrm{Re}(\bar{v}\,\partial_t v)) - 2t[A, V_2] * |v|^2 \\ &= 2iV_2 * (\mathrm{Im}(\bar{v}Kv)) - 2t[A, V_2] * |v|^2. \end{aligned}$$

We now let L act on Ku_j,

$$\begin{aligned} LKu_j &= KLu_j + 4itLu_j = KF_j(u_j) + 4itF_j(u_j) \\ &= (V_{1,j} + V_2 * |u_j|^2)\, Ku_j \\ &\qquad + [K, V_{1,j} + V_2 * |u_j|^2]\, u_j + 4itF_j(u_j) \\ &= (V_{1,j} + V_2 * |u_j|^2)\, Ku_j + 2i(V_2 * (\mathrm{Im}(\overline{u_j}\, Ku_j)))u_j \\ &\qquad + 4it(\tilde{V}_{1,j} + \tilde{V}_2 * |u_j|^2)u_j, \end{aligned} \tag{3.44}$$

where

$$\tilde{V}_{1,j} = V_{1,j} + (i/2)[A, V_{1,j}] = V_{1,j} + (\tfrac{1}{2})x \cdot \nabla V_{1,j},$$
$$\tilde{V}_2 = V_2 + (i/2)[A, V_2] = V_2 + (\tfrac{1}{2})x \cdot \nabla V_2.$$

On the other hand, we have

$$\begin{aligned}(Ku_j)(t) &= U(t-t_0)(Ku_j)(t_0) + \int_{t_0}^{t} \frac{d}{ds}(U(t-s)\, Ku_j(s))\, ds \\ &= U(t)\, U(-t_0)(Ku_j)(t_0) \\ &\quad - i(G_{t_0} LKu_j)(t), \qquad t, t_0 \in \mathbb{R}. \end{aligned} \tag{3.45}$$

Using (3.44) and (3.45), we estimate $|||Ku_j|||_{X(I;q)}$, where I is as in Lemma 2.8. We put (3.44) into (3.45), and use Lemmas 2.7–2.9 to obtain

$$\begin{aligned}|||Ku_j|||_{X(I;q)} &\leqslant C_7 \|Ku_j(t_0)\| + C_7 a^\eta (1 + \|\phi_j\|^2)\, |||Ku_j|||_{X(I;q)} \\ &\quad + C_7 a^\eta (1 + \|\phi_j\|^2)(1 + |t_0|^2)\, |||u_j|||_{X(I;q)}, \end{aligned} \tag{3.46}$$

where $C_7 = C(q)$. Since $(Ku_j)(0) = |x|^2 \phi_j$, we obtain the second part of (3.42) by the iterative use of (3.46) as in the proofs of Lemmas 3.1–3.2. We next estimate $|||J^2 u_j|||_{2,\infty;I_T}$, where $I_T = [-T, T]$, $T > 0$. From the relation $J^2 u_j = Ku_j - 2t^2 Lu_j = Ku_j - 2t^2 F(u_j)$, it follows that

$$|||J^2 u_j|||_{2,\infty;I_T} \leqslant |||Ku_j|||_{2,\infty;I_T} + 2 \sup_{|t| \leqslant T} t^2 \|F_j(u_j(t))\|. \tag{3.47}$$

Ket $q \in \Gamma$ and $\delta = \delta(q)$. By (2.11) we have

$$\begin{aligned}&\sup_{|t| \leqslant T} t^2 \|V_{1,j} u_j(t)\| \\ &\qquad \leqslant CT^2 \|\phi_j\| + CT^{2-2\delta} \|\phi_j\|^{1-\delta}\, |||J^2 u_j|||^{\delta}_{2,\infty;I_T}. \end{aligned} \tag{3.48}$$

By (2.15) and Lemma 2.1 we have

$$\begin{aligned}&\sup_{|t| \leqslant T} t^2 \|(V * |u_j|^2)\, u_j(t)\| \\ &\qquad \leqslant CT^2 \|\phi_j\|^3 + CT^{2-2\delta} \|\phi_j\|^{3-\delta}\, |||J^2 u_j|||^{\delta}_{2,\infty;I_T}. \end{aligned} \tag{3.49}$$

Since $\delta < 1$, we conclude from (3.47)–(3.49) that $|||J^2 u_j|||_{2,\infty;I_T}$ is bounded by $C(T, q, \sup_{j \in \mathbb{N}} \|\phi_j\|_{0,2})$. Therefore the first part of (3.42) follows from (3.24) and the inequality

$$\begin{aligned}\|J\psi\| &= \|xU(-t)\psi\| \\ &\leqslant \|\, |x|^2\, U(-t)\psi\|^{1/2}\, \|U(-t)\psi\|^{1/2} \\ &= \|J^2\psi\|^{1/2}\, \|\psi\|^{1/2}. \end{aligned} \tag{3.50}$$

We turn to (3.43). Let $v_{jk} = u_j - u_k$. Using (3.44)–(3.45), we write

$$Kv_{jk} = U(\cdot)\, U(-t_0)(Kv_{jk})(t_0) - iG_{t_0} LKv_{jk}, \tag{3.51}$$

with

$$\begin{aligned} LKv_{jk} = {} & VKv_{jk} + (V_{1,j} - V_{1,k})\, Ku_k + (V * |u_j|^2)\, Kv_{jk} \\ & + (V_2 * (\overline{v_{jk}} u_j + \overline{u_k} v_{jk}))\, Ku_k \\ & + 2i(V_2 * (\mathrm{Im}(\overline{v_{jk}} Ku_j)))u_j \\ & + 2i(V_2 * (\mathrm{Im}(\overline{u_k} Kv_{jk})))u_j \\ & + 2i(V_2 * (\mathrm{Im}(\overline{u_k} Ku_k)))v_{jk} \\ & + 4it\tilde{V}_{1,j} v_{jk} + 4it(\tilde{V}_{1,j} - \tilde{V}_{1,k})u_k \\ & + 4it(\tilde{V}_2 * |u_j|^2)v_{jk} \\ & + 4it(\tilde{V}_2 * (\overline{v_{jk}} u_j + \overline{u_k} v_{jk}))u_k. \end{aligned} \tag{3.52}$$

By Lemmas 2.7–2.9, we obtain from (3.51) to (3.52)

$$\begin{aligned} |||Kv_{jk}|||_{X(I;q)} \leqslant {} & C_8\, \|Kv_{jk}(t_0)\| \\ & + C_8 a^\eta (1 + \|\phi_j\|^2 + \|\phi_k\|^2)\, |||Kv_{jk}|||_{X(I;q)} \\ & + C_8 a^\eta (1 + |t_0|) \\ & \times \left(1 + \sum_{l=j,k} (\|\phi_l\|^2 + |||Ku_l|||^2_{2,\infty;I})\right) |||v_{jk}|||_{X(I;q)} \\ & + C_8 a^\eta\, |1/j - 1/k|^{\gamma_1/m} \\ & \times (1 + |t_0|)(|||Ku_k|||_{X(I;q)} + |||u_k|||_{X(I;q)}), \end{aligned} \tag{3.53}$$

where $C_8 = C(q)$. The second part of (3.43) follows from (3.42) and (3.53) in the same way as before. It remains to estimate $|||J^2 v_{jk}|||_{2,\infty;I_T}$. We write

$$\begin{aligned} J^2 v_{jk} = {} & Kv_{jk} - 2t^2(V_{1,j} v_{jk} + (V_{1,j} - V_{1,k})u_k \\ & + (V_2 * |u_j|^2)v_{jk} + (V_2 * (\overline{v_{jk}} u_j + \overline{u_k} v_{jk}))u_k). \end{aligned} \tag{3.54}$$

By Lemmas 2.8–2.9, we obtain from (3.54)

$$\begin{aligned} |||J^2 v_{jk}|||_{2,\infty;I} \leqslant {} & |||Kv_{jk}|||_{2,\infty;I} \\ & + 2 \sup_{t \in I} t^2(\|V_{1,j} v_{jk}\| + \|(V_{1,j} - V_{1,k})u_k\|) \\ & + C_9(1 + |t_0|^2)(\|\phi_j\|^2 + \|\phi_k\|^2)\, |||v_{jk}|||_{X(I;q)}, \end{aligned}$$

where $C_9 = C(q)$. In the same way as in the derivation of (3.48),

$$\begin{aligned} 2 \sup_{t \in I} t^2(\| V_{1,j} v_{jk} \| + \|(V_{1,j} - V_{1,k}) u_k \|) &\leqslant C_{10}(1 + |t_0|^2) \, |||v_{jk}|||_{2,\infty;I} \\ &\quad + C_{10} \sup_{t \in I} ((t^2 \, \|v_{jk}(t)\|)^{1-\delta} \, \|J^2 v_{jk}(t)\|^{\delta}) \\ &\quad + C_{10} \, |1/j - 1/k|^{\gamma_1/m} \, ((1 + |t_0|^2) \, \|\phi_k\| \\ &\quad + \sup_{t \in I} ((t^2 \, \|\phi_k\|)^{1-\delta} \, \|J^2 u_k(t)\|^{\delta})), \end{aligned} \tag{3.55}$$

where $C_{10} = C(q)$. Since $\delta < 1$, the R.H.S. of (3.55) is bounded by

$$\begin{aligned} &C(1 + |t_0|^2) \, |||v_{jk}|||_{2,\infty;I} + (\tfrac{1}{2}) \, |||J^2 v_{jk}|||_{2,\infty;I} \\ &\quad + C \, |1/j - 1/k|^{\gamma_1/m} \, ((1 + |t_0|^2) \, \|\phi_k\| + |||J^2 u_k|||_{2,\infty;I}), \end{aligned}$$

so that $|||J^2 v_{jk}|||_{2,\infty;I}$ is estimated by

$$\begin{aligned} &2 \, |||K v_{jk}|||_{2,\infty;I} + C(1 + |t_0|^2)(1 + \|\phi_j\|^2 + \|\phi_k\|^2) \, |||v_{jk}|||_{X(I;q)} \\ &\quad + C \, |1/j - 1/k|^{\gamma_1/m} \, ((1 + |t_0|^2) \, \|\phi_k\| + |||J^2 u_k|||_{2,\infty;I}). \end{aligned}$$

This leads to the global estimate

$$\begin{aligned} |||J^2 v_{jk}|||_{2,\infty;I_T} &\leqslant 2 \, |||K v_{jk}|||_{2,\infty;I_T} + C(T, q, \sup_{j \in \mathbb{N}} \|\phi_j\|_{0,2}) \\ &\quad \times (|||v_{jk}|||_{X(I;q)} + |1/j - 1/k|^{\gamma_1/m}), \end{aligned}$$

for $T > 0$, which implies the first part of (3.43). Q.E.D.

LEMMA 3.4. *Let $l \in \mathbb{N}$. Let $\phi \in H^{0,l}$ and let $\{\phi_j\}$ be a sequence in $\mathscr{S}$ such that $\phi_j \to \phi$ in $H^{0,l}$ as $j \to \infty$. Let $u_j \in C^1(\mathbb{R}; \mathscr{S})$ be the solution of (#) with $F_j(u_j)$ replaced by $(V_2 * |u_j|^2) u_j$. Let $q \in \Gamma$. Then for any $T > 0$, there exists a constant $C_T = C(T, q, l, \sup_{j \in \mathbb{N}} \|\phi_j\|_{0,l})$ such that*

$$\sup_{j \in \mathbb{N}} |||u_j|||_{X_l(T;q)} \leqslant C_T, \tag{3.56}$$

$$|||u_j - u_k|||_{X_l(T;q)} \leqslant C_T \cdot \|\phi_j - \phi_k\|_{0,l}, \qquad \text{for } j, k \in \mathbb{N}. \tag{3.57}$$

Proof. We use induction on $l \in \mathbb{N} \cup \{0\}$. For $l = 0$, the lemma reduces to Lemma 3.1. Now let $l \geqslant 1$ and assume that (3.56)–(3.57) hold for $l - 1$.

It is sufficient to prove

$$\sup_{j\in\mathbb{N}}\sum_{|\alpha|=l}|||J^{\alpha}u_j|||_{X(I;q)}\leqslant C_T, \tag{3.58}$$

$$\sum_{|\alpha|=l}|||J^{\alpha}(u_j-u_k)|||_{X(I;q)}\leqslant C_T\cdot\|\phi_j-\phi_k\|_{0,l}, \qquad j,k\in\mathbb{N}. \tag{3.59}$$

Let t_0, a, and I be as in Lemma 2.8. From (3.26) we have

$$\begin{aligned}J^{\alpha}u_j &= U(\cdot)\,U(-t_0)(J^{\alpha}u_j)(t_0)-iG_{t_0}(V_2*|u_j|^2)\,J^{\alpha}u_j\\ &\quad -iG_{t_0}(V_2*(\overline{u_j}J^{\alpha}u_j+(-1)^{|\alpha|}\,u_j\,\overline{J^{\alpha}u_j}))u_j\\ &\quad -i\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \alpha_1,\alpha_2,\alpha_3\neq\alpha}}\frac{(-1)^{|\alpha_1|}\cdot\alpha!}{\prod_{k=1}^{3}\alpha_k!}\\ &\quad \times G_{t_0}(V_2*(\overline{J^{\alpha_1}u_j}\,J^{\alpha_2}u_j))\,J^{\alpha_3}u_j, \qquad |\alpha|=l.\end{aligned} \tag{3.60}$$

By Lemmas 2.7–2.8, we obtain from (3.60)

$$\begin{aligned}|||J^{\alpha}u_j|||_{X(I;q)} &\leqslant C_{11}\,\|J^{\alpha}u_j(t_0)\|+C_{11}a^{\eta}\,\|\phi_j\|^2\,|||J^{\alpha}u_j|||_{X(I;q)}\\ &\quad +C_{11}a^{\eta}\sum_{|\beta_1+\beta_2+\beta_3|\leqslant l-1}\\ &\quad \times\prod_{k=1}^{3}|||J^{\beta_k}u_j|||_{X(I;q)}, \qquad |\alpha|=l,\end{aligned} \tag{3.61}$$

where $C_{11}=C(q,l)$. The second term in the R.H.S. of (3.61) is treated as in the proofs of Lemmas 3.1–3.3, while the third term is controlled by using the induction hypothesis. Repeating this process by the same length of intervals, we obtain (3.58). To prove (3.59), we write equations (3.60) for $J^{\alpha}u_j$ and for $J^{\alpha}u_k$, subtract the results, and use Lemmas 2.7–2.8, (3.58), with the induction hypothesis. Q.E.D.

Notation 4. Let $v\in\mathscr{X}$ and let $\{v_j\}$ be a sequence in $\mathscr{X}$. We write $v_j\to v$ in $\mathscr{X}$ as $j\to\infty$, if $v_j\to v$ in $C(\mathbb{R};L^2)$ and in $L^{\theta(q)}_{\mathrm{loc}}(\mathbb{R};L^q)$ for any $q\in\Gamma$, as $j\to\infty$.

Proof of Theorem 1. Let $\{\phi_j\}$ and $\{u_j\}$ be as in Lemma 3.1. It follows from Lemma 3.1 that there exists $u\in\mathscr{X}$ such that $u_j\to u$ in $\mathscr{X}$ as $j\to\infty$. Then the conservation of L^2-norm follows from (3.22), and therefore $u\in C_b(\mathbb{R};L^2)$. As in the proofs of Lemmas 2.8–2.9, we find $F(u)\in L^{\infty}(\mathbb{R};L^2)+\bigcap_{q\in\Gamma}L^{\theta'}_{\mathrm{loc}}(\mathbb{R};L^{q'})$, so that $GF(u)\in\mathscr{X}$. Moreover, an argument similar to the proof of Lemma 3.1 shows that $GF_j(u_j)\to GF(u)$ in $\mathscr{X}$ as $j\to\infty$. Taking the limit $j\to\infty$ in (3.21), we have (*) as an identity in $\mathscr{X}_b$, since u, $U\phi \in C_b(\mathbb{R}; L^2)$. We now prove the uniqueness of solutions in the space $C(\mathbb{R}; L^2) \cap L^\theta_{\text{loc}}(\mathbb{R}; L^q)$ for all $q \in \Gamma$. Let u, $v \in C(\mathbb{R}; L^2) \cap L^\theta_{\text{loc}}(\mathbb{R}; L^q)$ satisfy the integral equation. Define $\tilde{I} = \{t \in \mathbb{R};\ u(t) = v(t) \text{ in } L^2\}$. We show that $\tilde{I} = \mathbb{R}$. It is enough to show that $\tilde{I}$ is an open subset of $\mathbb{R}$. Let $t_0 \in \tilde{I}$, $I(a) = [t_0 - a,\ t_0 + a]$ for $0 < a \leqslant 1$, and let $\rho = \max(|||u|||_{2,\infty;I(1)}, |||v|||_{2,\infty;I(1)})$. From Lemmas 2.8–2.9 and the relation

$$\begin{aligned} u - v &= U(\cdot)\, U(-t_0)(u(t_0) - v(t_0)) - iG_{t_0}(F(u) - F(v)) \\ &= -iG_{t_0} V_1(u - v) - iG_{t_0}((V_2 * |u|^2)u - (V_2 * |v|^2)v), \end{aligned}$$

it follows that

$$|||u - v|||_{X(I(a);q)} \leqslant C(q)\, a^\eta (1 + \rho^2)\, |||u - v|||_{X(I(a);q)}.$$

This implies $(t_0 - a,\ t_0 + a) \subset \tilde{I}$ for some $a > 0$, as was to be shown. We have thus proved parts (1) and (2). We next prove part (3). In view of (1)–(2), it suffices to show $F(u) \in C_b(\mathbb{R}; H^{-2,0})$. Let $\psi \in H^{2,0}$. By Lemmas 2.4 and 2.1, we have

$$\begin{aligned} |(V_1 u(t), \psi)| &\leqslant \|u(t)\|\ \|V_1\psi\| \leqslant C\, \|\phi\|\ \|\psi\|_{2,0}, \qquad t \in \mathbb{R}, \\ |(V_1(u(t) - u(\tau)), \psi)| &\leqslant C\, \|u(t) - u(\tau)\|\ \|\psi\|_{2,0}, \qquad t, \tau \in \mathbb{R}. \end{aligned}$$

By Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} &|((V_2 * |u|^2)\, u(t), \psi)| \\ &\quad\leqslant \|u(t)\|^2\ \|V_2 * (\overline{u(t)}\, \psi)\|_\infty \\ &\quad\leqslant C\, \|\phi\|^2\, (\|\overline{u(t)}\, \psi\|_1 + \|\overline{u(t)}\, \psi\|_{1/(1/2 + 1/r)}) \\ &\quad\leqslant C\, \|\phi\|^2\, \|u(t)\|\ (\|\psi\| + \|\psi\|_r) \leqslant C\, \|\phi\|^3\ \|\psi\|_{2,0}, \qquad t \in \mathbb{R}, \end{aligned}$$

with r satisfying $\max(\frac{1}{2} - 2/n, 0) < 1/r < \frac{1}{2} - \max(\gamma_2, \gamma_3)/n$,

$$\begin{aligned} &|((V_2 * |u|^2)\, u(t) - (V_2 * |u|^2)\, u(\tau), \psi)| \\ &\qquad\leqslant C\, \|u(t) - u(\tau)\|\ \|\phi\|^2\ \|\psi\|_{2,0}, \qquad t, \tau \in \mathbb{R}. \end{aligned}$$

These estimates prove that $F(u) \in C_b(\mathbb{R}; H^{-2,0})$, as required. It remains to prove part (4). Let H and H_j be the self-adjoint operators defined by $H = -(\frac{1}{2})\,\Delta + V_1$, $H_j = -(\frac{1}{2})\,\Delta + V_{1,j}$, with the domains $D(H) = D(H_j) = H^{2,0}$, respectively. By Proposition 3.1, we have $u_j(t) = \exp(-itH_j)\phi_j$. In a way similar to (2.23), for $\psi \in H^{2,0}$ and $m > m_0$,

$$\begin{aligned} \|H_j\psi - H\psi\| &= \|V_{1,j}\psi - V\psi\| \\ &\leqslant Cj^{-\gamma_1/m}\, \|V_1^{1+1/m}\psi\| \to 0 \qquad \text{as} \quad j \to \infty. \end{aligned}$$

This gives

$$
\begin{aligned}
&\|(H_j+i)^{-1}-(H+i)^{-1}\|_{\mathscr{L}(L^2;L^2)}\\
&\quad=\|(H_j+i)^{-1}(H-H_j)(H+i)^{-1}\|_{\mathscr{L}(L^2;L^2)}\\
&\quad\leqslant\|(H-H_j)(H+i)^{-1}\|_{\mathscr{L}(L^2;L^2)}\to 0 \qquad \text{as} \quad j\to\infty.
\end{aligned}
$$

Therefore from the Trotter–Kato theorem, it follows that for $T>0$,

$$\sup_{|t|\leqslant T}\|\exp(-itH_j)\phi-\exp(-itH)\phi\|\to 0 \qquad \text{as} \quad j\to\infty.$$

Thus we obtain

$$
\begin{aligned}
&\sup_{|t|\leqslant T}\|u(t)-\exp(-itH)\phi\|\\
&\quad\leqslant\sup_{|t|\leqslant T}\|u(t)-u_j(t)\|+\sup_{|t|\leqslant T}\|u_j(t)-\exp(-itH_j)\phi\|\\
&\qquad+\sup_{|t|\leqslant T}\|\exp(-itH_j)\phi-\exp(-itH)\phi\|\\
&\quad=\sup_{|t|\leqslant T}\|u(t)-u_j(t)\|+\|\phi_j-\phi\|\\
&\qquad+\sup_{|t|\leqslant T}\|\exp(-itH_j)\phi-\exp(-itH)\phi\|\to 0 \qquad \text{as} \quad j\to\infty.
\end{aligned}
$$

Q.E.D.

LEMMA 3.5. *Let $\phi\in H^{0,1}$ and let $\{\phi_j\}$ be a sequence in $\mathscr{S}$ such that $\phi_j\to\phi$ in $H^{0,1}$ as $j\to\infty$. Let $u_j\in C^1(\mathbb{R};\mathscr{S})$ be the solution of $(\#)$. Then there exists a constant $C=C(\sup_{j\in\mathbb{N}}\|\phi_j\|_{0,1})$ such that*

$$\sup_{j\in\mathbb{N}}\|Ju_j(t)\|\leqslant C\cdot(1+|t|), \qquad t\in\mathbb{R}.$$

Proof. Let $\alpha_j(t)=\|Ju_j\|^2+2t^2(V_{1,j}u_j,\ u_j)+t^2((V_2*|u_j|^2)u_j,\ u_j)$. A direct calculation shows

$$\frac{d}{dt}\alpha_j(t)=4t(\tilde{V}_{1,j}u_j,u_j)+2t((\tilde{V}_2*|u_j|^2)u_j,u_j), \qquad t\in\mathbb{R}, \tag{3.62}$$

where $\tilde{V}_{1,j}$ and $\tilde{V}_2$ are as in (3.44). Let $q\in\Gamma$. Let $\varepsilon>0$ be sufficiently small. In the same way as in the proof of Lemma 2.4, we have

$$
\begin{aligned}
&|(V_{1,j}u_j,u_j)|\\
&\quad\leqslant C(\varepsilon)\|\phi_j\|^2+\varepsilon t^{-2}\|Ju_j\|^2, \qquad t\in\mathbb{R}\backslash\{0\},
\end{aligned}
\tag{3.63}
$$

$$|(\tilde{V}_{1,j}u_j,u_j)|$$

$$\leqslant\begin{cases} C\|\phi_j\|^2+C|t|^{-2\delta}\|\phi_j\|^{2-2\delta}(1+\|Ju_j\|^2), & t\in\mathbb{R}\backslash\{0\}, \quad (3.64)\\ C(\varepsilon)\|\phi_j\|^2+\varepsilon t^{-2}\|Ju_j\|^2, & t\in\mathbb{R}\backslash\{0\}, \quad (3.65)\end{cases}$$

where $\delta=\delta(q)$. By Lemmas 2.1–2.2, we have

$$|((V_2 * |u_j|)u_j, u_j)| \leqslant C(\varepsilon)\sum_{k=2}^{3} \|\phi_j\|^{2(4-\gamma_k)/(2-\gamma_k)} + \varepsilon t^{-2}\|Ju_j\|^2, \qquad t\in\mathbb{R}\backslash\{0\}, \tag{3.66}$$

$$|((\tilde{V}_2 * |u_j|^2)u_j, u_j)| \leqslant \begin{cases} C\sum_{k=2}^{3} |t|^{-\gamma_k}\|\phi_j\|^{4-\gamma_k}(1+\|Ju_j\|^2), & t\in\mathbb{R}\backslash\{0\}, \quad (3.67)\\ C(\varepsilon)\sum_{k=2}^{3}\|\phi_j\|^{2(4-\gamma_k)/(2-\gamma_k)} + \varepsilon t^{-2}\|Ju_j\|^2, & t\in\mathbb{R}\backslash\{0\}. \quad (3.68)\end{cases}$$

By (3.63), (3.66) and by the definition of $\alpha_j(t)$, we obtain

$$\|Ju_j(t)\|^2 \leqslant 2\alpha_j(t) + C(\sup_{j\in\mathbb{N}}\|\phi_j\|)t^2, \qquad t\in\mathbb{R}. \tag{3.69}$$

By (3.62), (3.64), (3.67), we obtain

$$|\alpha_j(t)| \leqslant \|x\phi_j\|^2 + \left|\int_0^t \frac{d}{ds}\alpha_j(s)\,ds\right|$$
$$\leqslant \|x\phi_j\|^2 + C_{12}(1+t^2) + \left|\int_0^t g(s)\,\|Ju_j\|^2\,ds\right|, \qquad t\in\mathbb{R}, \tag{3.70}$$

where $C_{12}=C(\sup_{j\in\mathbb{N}}\|\phi_j\|)$ and $g(s)=|s|^{1-2\delta}+\sum_{k=2}^{3}|s|^{1-\gamma_k}$. Gronwall's inequality applied to (3.69)–(3.70) results in

$$\|Ju_j(t)\|^2 \leqslant C(\sup_{j\in\mathbb{N}}\|\phi_j\|_{0,1})\cdot(1+t^2) \times \exp(C(\sup_{j\in\mathbb{N}}\|\phi_j\|)\cdot(1+t^2)). \tag{3.71}$$

We now improve this estimate as regards the growth order in time. By (3.62), (3.65), (3.68), we have

$$\left|\frac{d}{dt}(t^{-1}\alpha_j(t))\right| \leqslant C(\sup_{j\in\mathbb{N}}\|\phi_j\|_{0,1}), \qquad t\in\mathbb{R}\backslash\{0\},$$

and therefore

$$|\alpha_j(t)| \leqslant |t|\max(|\alpha_j(1)|, |\alpha_j(-1)|) + C(\sup_{j\in\mathbb{N}}\|\phi_j\|)t^2, \qquad |t|\geqslant 1. \tag{3.72}$$

The lemma then follows from (3.69), (3.71), and (3.72). Q.E.D.

PROPOSITION 3.2. *Let $\phi \in H^{0,1}$ and let $u \in \mathscr{X}_b$ the solution of* (*). *Then*:

(1) *$Ju \in C(\mathbb{R}; L^2)$. Moreover, Ju has the estimate*

$$\|Ju(t)\| \leqslant C(\|\phi\|_{0,1}) \cdot (1+|t|), \qquad t \in \mathbb{R}. \tag{3.73}$$

(2) *Let $\alpha(t) = \|Ju(t)\|^2 + 2t^2(V_1 u(t),\ u(t)) + t^2((V_2 * |u|^2)\, u(t),\ u(t))$. Then α is absolutely continuous on $\mathbb{R}$ and satisfies the identity*

$$\alpha(t) = \alpha(\tau) + 4\int_\tau^t s\{(\tilde{V}_1 u, u) + \tfrac{1}{2}((\tilde{V}_2 * |u|^2)u, u)\}\, ds, \qquad t, \tau \in \mathbb{R}. \tag{3.74}$$

(3) *Let $\{\phi_j\}$ be a sequence in $\mathscr{S}$ such that $\phi_j \to \phi$ in $H^{0,1}$ as $j \to \infty$. Let u_j be the solution of (#). Then $U^{-1}u_j \to U^{-1}u$ in $C(\mathbb{R}; H^{0,1})$ as $j \to \infty$.*

Proof. Let $\{\phi_j\}$ and $\{u_j\}$ be as in part (3). It follows from Lemma 3.5 and Theorem 1 that $Ju \in C_w(\mathbb{R}; L^2)$ and Ju has the estimate (3.73). Moreover, $Ju_j(t) \to Ju(t)$ weakly in L^2 uniformly on compact t-intervals as $j \to \infty$. Namely, for any $\psi \in L^2$ and $T > 0$,

$$\sup_{|t| \leqslant T} |(Ju_j(t) - Ju(t), \psi)| \to 0 \qquad \text{as} \quad j \to \infty. \tag{3.75}$$

We proceed to part (2). It suffices to prove, for any t_0, $t_1 \in \mathbb{R}$,

$$\alpha(t_1) \leqslant \alpha(t_0) + 4\int_{t_0}^{t_1} \tilde{\alpha}(t)\, dt, \tag{3.76}$$

where $\tilde{\alpha}(t) = t(\tilde{V}_1 u(t),\ u(t)) + (t/2)((\tilde{V}_2 * |u|^2)\, u(t),\ u(t))$. (Estimates similar to (3.63)–(3.68) imply that every term in $\alpha(t)$ or in $\tilde{\alpha}(t)$ is finite for all $t \in \mathbb{R}$.)

In order to prove (3.76), we approximate $u(t_0)$ by a sequence $\{\psi_j\}$ in $\mathscr{S}$ such that $U(-t_0)\psi_j \to U(-t_0)\, u(t_0)$ in $H^{0,1}$ as $j \to \infty$, and we consider the integral equation

$$v_j(t) = U(t - t_0)\psi_j - i(G_{t_0} F_j(v_j))(t), \qquad t \in \mathbb{R}. \tag{3.77}$$

Let $q \in \Gamma$. We already know that (3.77) has a unique solution $v_j \in C^1(\mathbb{R}; \mathscr{S})$ satisfying

$$\|v_j(t)\| = \|\psi_j\|, \qquad t \in \mathbb{R}, \tag{3.78}$$

$$\begin{aligned} &\|Jv_j(t)\|^2 + 2t^2(V_{1,j}v_j(t), v_j(t)) + t^2((V_2 * |v_j|^2)\, v_j(t), v_j(t)) \\ &\quad = \|J(t_0)\psi_j\|^2 + 2t_0^2(V_{1,j}\psi_j, \psi_j) + t_0^2((V_2 * |\psi_j|^2)\psi_j, \psi_j) \\ &\qquad + 4\int_{t_0}^t s\left\{(\tilde{V}_{1,j}v_j, v_j) + \frac{1}{2}((\tilde{V}_2 * |v_j|^2)v_j, v_j)\right\} ds, \qquad t \in \mathbb{R}. \end{aligned} \tag{3.79}$$

Moreover, for any $T>0$, $\{v_j\}$ is a Cauchy sequence in $X([t_0-T, t_0+T]; q)$, and $\{J_{vj}\}$ is a bounded sequence in $C_b([t_0-T, t_0+T]; L^2)$. The same argument as in the proof of Theorem 1 shows that there exists $v \in \mathscr{X}_b$ satisfying $v_j \to v$ in $\mathscr{X}$ as $j \to \infty$, and

$$v(t) = U(t-t_0)\, u(t_0) - i(G_{t_0} F(v))(t), \qquad t \in \mathbb{R}.$$

The uniqueness of the integral equation implies $u = v$. Consequently, $v_j \to v$ in $\mathscr{X}$ as $j \to \infty$, and $Jv_j(t) \to Ju(t)$ weakly in L^2 uniformly on compact t-intervals as $j \to \infty$. We now derive (3.76) from (3.79) by a limiting argument. We consider the convergence of each term in (3.79). Let $m > m_0$. By (2.8), (2.23), and (3.78), we obtain

$$\begin{aligned}
&t^2\, |(V_{1,j} v_j, v_j) - (V_1 u, u)| \\
&\quad \leqslant t^2\, |((V_{1,j} - V_1) v_j, v_j)| \\
&\qquad + t^2\, |(V_1(v_j - u), v_j)| + t^2\, |(V_1 u, v_j - u)| \\
&\quad \leqslant Cj^{-\gamma_1/m}(|t|\, \|\psi\| + |t|^{1-\delta}\, \|\psi_j\|^{1-\delta}\, \|Jv_j\|^{\delta})^2 \\
&\qquad + Ct^2(\|\psi_j\| + \|\phi\|)\, \|v_j - u\| \\
&\qquad + C\, |t|^{2-2\delta}\, (\|\psi_j\| + \|\phi\|)^{1-\delta}\, \|v_j - u\|^{1-\delta} \\
&\qquad \times (\|Jv_j\| + \|Ju\|)^{2\delta},
\end{aligned}$$

and therefore, for any $T > 0$,

$$\sup_{|t-t_0| \leqslant T} t^2\, |(V_{1,j} v_j, v_j) - (V_1 u, u)| \to 0 \qquad \text{as} \quad j \to \infty. \tag{3.80}$$

Similarly,

$$\begin{aligned}
&\sup_{|t-t_0| \leqslant T} \left| \int_{t_0}^{t} s((\tilde{V}_{1,j} v_j, v_j) - (\tilde{V}_1 u, u))\, ds \right| \\
&\quad \leqslant Cj^{-\gamma_1/m}(T\, \|\psi_j\| + T^{1-\delta}\, \|\psi_j\|^{1-\delta} \sup_{|t-t_0| \leqslant T} \|Jv_j\|^{\delta})^2 \\
&\qquad + CT^2(\|\psi_j\| + \|\phi\|) \sup_{|t-t_0| \leqslant T} \|v_j - u\| \\
&\qquad + CT^{2-2\delta}(\|\psi_j\| + \|\phi\|)^{1-\delta} \\
&\qquad \times \sup_{|t-t_0| \leqslant T} (\|v_j - u\|^{1-\delta}\, (\|Jv_j\| + \|Ju\|)^{2\delta}) \\
&\quad \to 0 \qquad \text{as} \quad j \to \infty.
\end{aligned} \tag{3.81}$$

By Lemmas 2.1–2.2, we obtain

$$\begin{aligned}
&\sup_{|t-t_0|\leqslant T} t^2 \, |((V_2 * |v_j|^2 v_j, v_j) - ((V_2 * |u|^2)u, u)| \\
&\quad \leqslant C \sup_{|t-t_0|\leqslant T} \left(\|v_j - u\| \sum_{k=2}^{3} |t|^{2-\gamma_k} \right. \\
&\qquad \left. \times (\|\psi_j\| + \|\phi\|)^{3-\gamma_k} (\|Jv_j\| + \|Ju\|)^{\gamma_k} \right) \\
&\quad \to 0 \qquad \text{as} \quad j \to \infty. \qquad (3.82)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sup_{|t-t_0|\leqslant T} \left| \int_{t_0}^{t} s(((\tilde{V}_2 * |v_j|^2)v_j, v_j) - ((\tilde{V}_2 * |u|^2)u, u)) \, ds \right| \\
&\quad \leqslant C \sum_{k=2}^{3} T^{2-\gamma_k}(\|\psi_j\| + \|\phi\|)^{3-\gamma_k} \\
&\qquad \sup_{|t-t_0|\leqslant T} (\|v_j - u\| \, (\|Jv_j\| + \|Ju\|)^{\gamma_k}) \\
&\quad \to 0 \qquad \text{as} \quad j \to \infty. \qquad (3.83)
\end{aligned}$$

Taking the limit inferior $j \to \infty$ in (3.79), we obtain (3.76), since $Jv_j(t_1) \to Ju(t_1)$ weakly in L^2 as $j \to \infty$. We have thus proved the identity (3.74). Moreover, estimates similar to (3.81), (3.83) imply that $\tilde{\alpha}$ is locally integrable on $\mathbb{R}$, so that α is absolutely continuous on $\mathbb{R}$. We also find from (3.80) and (3.82) that the maps $t \mapsto t^2(V_1 u(t), u(t))$ and $t \mapsto t^2((V_2 * |u|^2)\, u(t), u(t))$ are continuous on $\mathbb{R}$, and therefore $t \mapsto \|Ju(t)\|^2$ is continuous on $\mathbb{R}$. This gives $Ju \in C(\mathbb{R}; L^2)$. It remains to prove part (3). We claim that for any $T > 0$,

$$\sup_{|t|\leqslant T} | \, \|Ju_j(t)\|^2 - \|Ju(t)\|^2| \to 0 \qquad \text{as} \quad j \to \infty. \qquad (3.84)$$

To prove this, we write $\|Ju_j(t)\|^2 - \|Ju(t)\|^2$ as

$$\begin{aligned}
&\|x\phi_j\|^2 - \|x\phi\|^2 + 2t^2((V_{1,j}u_j, u_j) - (V_1 u, u)) + t^2(((V_2 * |u_j|^2)u_j, u_j) \\
&\quad - ((V_2 * |u|^2)u, u)) + 4 \int_0^t s((\tilde{V}_1 u, u) - (\tilde{V}_{1,j}u_j, u_j)) \, ds \\
&\quad + 2 \int_0^t s(((\tilde{V}_2 * |u|^2)u, u) - ((\tilde{V}_2 * u_j|^2)u_j, u_j)) \, ds.
\end{aligned}$$

Convergence (3.84) then follows by applying estimates similar to (3.80)–(3.83) to the R.H.S. of the above identity. In order to prove part (3), we suppose that this does not hold, and we deduce a contradiction. We already know by the proof of Theorem 1 that $u_j \to u$ in $C(\mathbb{R}; L^2)$ as $j \to \infty$. Then there exist T, $\varepsilon > 0$ and a subsequence denoted again by $\{Ju_j\}$, such that

$$\sup_{|t| \leqslant T} \|Ju_j(t) - Ju(t)\| \geqslant \varepsilon, \qquad \text{for all} \quad j \in \mathbb{N}.$$

We choose a sequence $\{t_j\} \subset [-T, T]$ such that

$$\|Ju_j(t_j) - Ju(t_j)\| \geqslant \varepsilon/2, \qquad \text{for all} \quad j \in \mathbb{N}. \tag{3.85}$$

Then there exists a subsequence $\{j_k\} \subset \mathbb{N}$ and $t_0 \in [-T, T]$ such that $j_1 < j_2 < \cdots < j_k \uparrow \infty$ and $t_{j_k} \to t_0$ as $k \to \infty$. By (3.75), for any $\psi \in L^2$,

$$\begin{aligned} |(Ju_{j_k}(t_{j_k}) - Ju(t_{j_k}), \psi)| &\leqslant \sup_{|t| \leqslant T} |(Ju_{j_k}(t) - Ju(t), \psi)| \\ &\to 0 \qquad \text{as} \quad j \to \infty. \end{aligned} \tag{3.86}$$

Now we write

$$\|Ju_{j_k}(t_{j_k}) - Ju(t_{j_k})\|^2 = I_{1,k} + I_{2,k},$$

where

$$\begin{aligned} I_{1,k} &= \|Ju_{j_k}(t_{j_k})\|^2 - \|Ju(t_{j_k})\|^2, \\ I_{2,k} &= 2\,\mathrm{Re}(Ju(t_{j_k}) - Ju_{j_k}(t_{j_k}), Ju(t_{j_k})). \end{aligned}$$

We obtain

$$\lim_{k \to \infty} I_{1,k} = 0$$

by (3.84),

$$\lim_{k \to \infty} I_{2,k} = 0$$

by (3.86) and by the fact that $Ju \in C(\mathbb{R}; L^2)$. Hence

$$\lim_{k \to \infty} \|Ju_{j_k}(t_{j_k}) - Ju(t_{j_k})\| = 0,$$

which contradicts (3.85). Q.E.D.

LEMMA 3.6. *Let $\phi \in H^{0,2}$ and let $\{\phi_j\}$ be a sequence in $\mathscr{S}$ such that $\phi_j \to \phi$ in $H^{0,2}$ as $j \to \infty$. Let $u_j \in C^1(\mathbb{R}; \mathscr{S})$ be the solution of (#). Let*

$\beta(t) = \|J^2 u_j(t) + 2t^2 F_j(u_j(t))\|^2 - 4t^2(V_2 * (\mathrm{Im}(\overline{u_j}\, Ju_j(t))), \quad \mathrm{Im}(\overline{u_j}\, Ju_j(t)))$. *Then β_j satisfies the identity*

$$\begin{aligned}\frac{d}{dt}\beta_j(t) = &-8t(\tilde{V}_2 * (\mathrm{Im}(\overline{u_j}\, Ju_j(t))), \mathrm{Im}(\overline{u_j}\, Ju_j(t)))\\ &+ 8t\,\mathrm{Re}(J^2 u_j(t) + 2t^2 F_j(u_j(t)), (\tilde{V}_{1,j} + \tilde{V}_2 * |u_j|^2)\, u_j(t))\\ &+ 4(V_2 * (\mathrm{Im}(\overline{u_j}\, J^2 u_j(t))), |Ju_j(t)|^2 + 2t^2 F_j(u_j(t))\,\overline{u_j(t)}), \quad t \in \mathbb{R}.\end{aligned} \tag{3.87}$$

Moreover, there exists a constant $C = C(\sup_{j\in\mathbb{N}} \|\phi_j\|_{0,2})$ such that

$$\sup_{j\in\mathbb{N}} \|J^2 u_j(t)\| \leqslant C(1+|t|)^{2(3-\gamma^0)/(2-\gamma^0)}, \qquad t \in \mathbb{R}. \tag{3.88}$$

If in addition $\lambda_2 = \lambda_3 = 0$, then there exists a constant C independent of $j \in \mathbb{N}$ such that

$$\|J^2 u_j(t)\| \leqslant C\, \|\phi_j\|_{0,2}\, (1+|t|)^2, \qquad t \in \mathbb{R},\ j \in \mathbb{N}. \tag{3.89}$$

Proof. Let $v_j(t) = S(-t)\, u_j(t)$ for $t \neq 0$. Then $v_j \in C^1(\mathbb{R}\backslash\{0\}; \mathscr{S})$ and v_j satisfies the differential equation (see [8]):

$$i\partial_t v_j + \frac{1}{2}\Delta v_j = \frac{1}{t} A v_j + F_j(v_j), \qquad t \in \mathbb{R}\backslash\{0\}. \tag{3.90}$$

Equation (3.87) can be obtained from (3.90). For the proof of (3.87), we refer the reader to J. Ginibre [16], since the original proof is rather complicated.

We now turn to (3.88)–(3.89). Since we already know (3.42), we prove (3.88)–(3.89) only for the case $|t| \geqslant 1$. From Lemmas 2.1, 2.2, and 3.5, it follows that

$$\begin{aligned}&|t^2(V_2 * (\mathrm{Im}(\overline{u_j}\, Ju_j)), \mathrm{Im}(\overline{u_j}\, Ju_j))|\\ &\qquad + |t^2(\tilde{V}_2 * (\mathrm{Im}(\overline{u_j}\, Ju_j)), \mathrm{Im}(\overline{u_j}\, Ju_j))|\\ &\qquad \leqslant C \sum_{k=2}^{3} |t|^{2-\gamma_k}\, \|Ju_j\|^{2+\gamma_k}\, \|\phi_j\|^{2-\gamma_k}\\ &\qquad \leqslant C(\sup_{j\in\mathbb{N}} \|\phi_j\|_{0,1})\, t^4.\end{aligned} \tag{3.91}$$

Then by the definition of β_j and by (3.47)–(3.49), we obtain

$$\|J^2 u_j(t)\|^2,\ \|K u_j(t)\|^2 \leqslant 2\beta_j(t) + C(\sup_{j\in\mathbb{N}} \|\phi_j\|_{0,1})\, t^4, \tag{3.92}$$

$$\|J^2 u_j(t)\|^2 \leqslant 2\, \|K u_j(t)\|^2 + C(\sup_{j\in\mathbb{N}} \|\phi_j\|_{0,1})\, t^4. \tag{3.93}$$

From (3.87), we get

$$\frac{d}{dt}(t^{-3}\beta_j(t)) = -3t^{-4}\|Ku_j\|^2$$
$$+12t^{-2}(V_2 * (\mathrm{Im}(\overline{u_j} Ju_j)), \mathrm{Im}(\overline{u_j} Ju_j))$$
$$-8t^{-2}(\tilde{V}_2 * (\mathrm{Im}(\overline{u_j} Ju_j)), \mathrm{Im}(\overline{u_j} Ju_j))$$
$$+4t^{-3}\,\mathrm{Im}(V_2 * (\overline{u_j} J^2u_j), |Ju_j|^2)$$
$$+8t^{-1}\,\mathrm{Im}(V_2 * (\overline{u_j} Ju_j), V_{1,j}|u_j|^2$$
$$+8t^{-1}\,\mathrm{Im}(V_2 * (\overline{u_j} J^2u_j), (V_2 * |u_j|^2)|u_j|^2)$$
$$+8t^{-2}\,\mathrm{Re}(Ku_j, (\tilde{V}_1 + \tilde{V}_2 * |u_j|^2)u_j). \tag{3.94}$$

We estimate the last four terms on the R.H.S. of (3.94). Let $\varepsilon > 0$ be sufficiently small. By Lemmas 2.1, 2.2, and 3.5, we have

$$|t^{-3}\,\mathrm{Im}(V_2 * (\overline{u_j} J^2u_j), |Ju_j|^2)|$$
$$\leqslant C\sum_{k=2}^{3} |t|^{\gamma_k}\|\phi_j\|^{1-\gamma_k/2}\|Ju_j\|^2\|J^2u_j\|^{1+\gamma_k/2}$$
$$\leqslant \varepsilon t^{-4}\|J^2u_j\|^2 + C(\sup_{j\in\mathbb{N}}\|\phi_j\|_{0,1}, \varepsilon)|t|^{4/(2-\gamma^0)}. \tag{3.95}$$

We claim that

$$|t^{-1}\,\mathrm{Im}(V_2 * (\overline{u_j} J^2u_j), V_{1,j}|u_j|^2)|$$
$$\leqslant \varepsilon t^{-4}\|J^2u_j\|^2 + C(\sup_{j\in\mathbb{N}}\|\phi_j\|_{0,1}, \varepsilon)|t|^{4/(2-\gamma^0)}. \tag{3.96}$$

To prove (3.96), we distinguish between two cases: (1) $n \leqslant 2$. (2) $n \geqslant 3$.

(1) When $n \leqslant 2$, we choose q so that $\max(\gamma_2, \gamma_3)/4 < \delta(q) < n/2$. Then by Hölder's inequality and Lemmas 2.1, 2.2, 3.5, we have

$$\|V_{1,j}|u_j|^2\|_{2n/(2n-\gamma_k)}$$
$$\leqslant C\cdot(\|u_j\|_q^2 + \|u_j\|_{4n/(2n-\gamma_k)}^2)$$
$$\leqslant C(\sup_{j\in\mathbb{N}}\|\phi_j\|_{0,1}), \qquad k = 2, 3,$$
$$|t^{-1}\,\mathrm{Im}(V_2 * (\overline{u_j} J^2u_j), V_{1,j}|u_j|^2)|$$
$$\leqslant C\sum_{k=2}^{3}|t|^{-1}\|\overline{u_j} J^2u_j\|_{2n/(2n-\gamma_k)}\|V_{1,j}|u_j|^2\|_{2n/(2n-\gamma_k)}$$
$$\leqslant C(\sup_{j\in\mathbb{N}}\|\phi_j\|_{0,1})|t|^{-1}\|J^2u_j\|,$$

from which (3.96) follows.

(2) When $n \geqslant 3$, we write

$$\begin{aligned}
&|t^{-1}\,\mathrm{Im}(V_2 * (\mathrm{Im}(\overline{u_j}\,J^2 u_j)),\ V_{1,j}\,|u_j|^2)| \\
&\qquad = |t\,\mathrm{Im}((\nabla V_2) * (\overline{v_j}\,\nabla v_j),\ V_{1,j}\,|v_j|^2)|.
\end{aligned}$$

Let $Q_\gamma = |x|^{-\gamma}$ for $0<\gamma<2$, so that $V_2 = \sum_{k=2}^{3} \lambda_k Q_{\gamma_k}$. Let $q \in \Gamma$. If $0<\gamma\leqslant 1$, we have by Lemmas 2.4, 2.6, and 3.5,

$$\begin{aligned}
&|t\,\mathrm{Im}((\nabla Q_\gamma) * (\overline{v_j}\,\nabla v_j),\ V_{1,j}\,|v_j|^2)| \\
&\qquad \leqslant \|t(\nabla Q_\gamma) * (\overline{v_j}\,\nabla v_j)\|_\infty\ \|V_{1,j}\,|v_j|^2\|_1 \\
&\qquad \leqslant C(\sup_{j\in\mathbb{N}} \|\phi_j\|_{0,1})\,|t|\ \|\Delta v_j\|^\gamma.
\end{aligned}$$

If $1<\gamma<2$, we have by Lemmas 2.1, 2.2, 2.4, and 3.5,

$$\begin{aligned}
&|t\,\mathrm{Im}((\nabla Q_\gamma) * (\overline{v_j}\,\nabla v_j),\ V_{1,j}\,|v_j|^2)| \\
&\qquad \leqslant \|t(\nabla Q_\gamma) * (\overline{v_j}\,\nabla v_j)\|_n\ \|v_j\|_{2n/(n-2)}\ \|V_{1,j} v_j\| \\
&\qquad \leqslant C(\sup_{j\in\mathbb{N}} \|\phi_j\|_{0,1})\,|t|\ \|\Delta v_j\|^{\gamma-1}\,(1+\|\Delta v_j\|^\delta).
\end{aligned}$$

These estimates prove (3.96).

We next estimate the last two terms on the R.H.S. of (3.94). By Lemmas 2.1, 2.2, and 3.5, we obtain

$$\begin{aligned}
&|t^{-1}\,\mathrm{Im}(V_2 * (\overline{u_j}\,J^2 u_j),\ (V_2 * |u_j|^2)\,|u_j|^2)| \\
&\qquad \leqslant t^{-1}\,\|(V_2 * (\overline{u_j}\,J^2 u_j))u_j\|\ \|(V_2 * |u_j|^2)u_j\| \\
&\qquad \leqslant C(\sup_{j\in\mathbb{N}} \|\phi_j\|_{0,1})\,|t|^{-1}\,\|J^2 u_j\| \\
&\qquad \leqslant \varepsilon\,|t|^{-4}\,\|J^2 u_j\|^2 + C(\sup_{j\in\mathbb{N}} \|\phi_j\|_{0,1}, \varepsilon)\,t^2. \qquad (3.97)
\end{aligned}$$

By Lemmas 2.1, 2.2, 2.4, 3.5, and (3.93), we obtain

$$\begin{aligned}
&|t^{-2}\,\mathrm{Re}(Ku_j,\ (\tilde{V}_{1,j} + \tilde{V}_2 * |u_j|^2)u_j)| \\
&\qquad \leqslant \varepsilon t^{-4}\,\|Ku_j\|^2 + C(\sup_{j\in\mathbb{N}} \|\phi_j\|_{0,1}, \varepsilon). \qquad (3.98)
\end{aligned}$$

From (3.91), (3.92), (3.95)–(3.98), it follows that

$$\left|\frac{d}{dt}(t^{-3}\beta_j(t))\right| \leqslant C(\sup_{j\in\mathbb{N}} \|\phi_j\|_{0,1})\,|t|^{4/(2-\gamma^0)},$$

which together with (3.92) gives (3.88). If $V_2 = 0$, then (3.89) follows from (3.94) and (3.98). Q.E.D.

PROPOSITION 3.3. *Let $\phi \in H^{0,2}$ and let $u \in \mathscr{X}_b$ be the solution of* (*). *Then*:

(1) *$J^2u \in C(\mathbb{R}; L^2)$ and J^2u has the estimate*

$$\|J^2u(t)\| \leqslant C(\|\phi\|_{0,2})(1+|t|)^{2(3-\gamma^0)/(2-\gamma^0)}, \qquad t \in \mathbb{R}. \tag{3.99}$$

If $\lambda_2 = \lambda_3 = 0$, then there exists a constant $C = C(n, \gamma_1, \lambda_1)$ such that

$$\|J^2u(t)\| \leqslant C\,\|\phi\|\,(1+|t|)^2, \qquad t \in \mathbb{R}.$$

(2) *Let $\tilde{K}(u) = J^2u + 2t^2F(u)$. Then $\tilde{K}(u) \in \mathscr{X}$ and $\tilde{K}(u)$ satisfies*

$$\tilde{K}(u) = U\,|x|^2\,\phi - iG\tilde{F}(u)$$

in $C(\mathbb{R}; L^2)$, with

$$\begin{aligned}\tilde{F}(u) = {} & (V_1 + V_2 * |u|^2)\,\tilde{K}(u) + 2i(V_2 * (\mathrm{Im}(\bar{u}\tilde{K}(u))))u \\ & + 4it(\tilde{V}_1 + \tilde{V}_2 * |u|^2)u\end{aligned}$$

in $C(\mathbb{R}; L^2) + \bigcap_{q \in \Gamma} L^{\theta(q)'}_{\mathrm{loc}}(\mathbb{R}; L^{q'})$.

(3) *Let $\beta(t) = \|(\tilde{K}(u))(t)\|^2 - 4t^2(V_2 * (\mathrm{Im}(\bar{u}Ju(t))), \mathrm{Im}(\bar{u}Ju(t)))$. Then β is absolutely continuous on $\mathbb{R}$ and β satisfies*

$$\begin{aligned}\beta(t) = \beta(t) + 8\int_\tau^t & s\{\mathrm{Re}(\tilde{K}(u), (\tilde{V}_1 + \tilde{V}_2 * |u|^2)u) \\ & - (\tilde{V}_2 * (\mathrm{Im}(\bar{u}Ju)), \mathrm{Im}(\bar{u}Ju))\}\,ds \\ & + 4\int_\tau^t (V_2 * (\mathrm{Im}(\bar{u}\tilde{K}(u))), |Ju|^2 \\ & + 2s^2(V_1 + V_2 * |u|^2)\,|u|^2)\,ds, \qquad t, \tau \in \mathbb{R}.\end{aligned} \tag{3.100}$$

(4) *Let $\{\phi_j\}$ be a sequence in $\mathscr{S}$ such that $\phi_j \to \phi$ in $H^{0,2}$ as $j \to \infty$. Let $u_j \in C^1(\mathbb{R}; \mathscr{S})$ be the solution of* (#). *Then $U^{-1}u_j \to U^{-1}u$ in $C(\mathbb{R}; H^{0,2})$, $Ku_j \to \tilde{K}(u)$ in $\mathscr{X}$, as $j \to \infty$.*

Proof. Let $\{\phi_j\}$ and $\{u_j\}$ be as in part (4). It follows from Lemma 3.3 that there exists $\hat{K}(u) \in \mathscr{X}$ such that $Ku_j \to \hat{K}(u)$ as $j \to \infty$. We also have $J^2u_j \to J^2u$ in $C(\mathbb{R}; L^2)$ as $j \to \infty$. Then part (1) follows from (3.88)–(3.89). In the same way as in the proof of Lemma 3.3, we find that for any $T > 0$,

$$\sup_{|t| \leqslant T} \|t^2F_j(u_j(t)) - t^2F(u(t))\| \to 0 \qquad \text{as} \quad j \to \infty.$$

Thus we have $\hat{K}(u)=J^2u+2t^2F(u)=\tilde{K}(u)$ as an identity in $C(\mathbb{R}; L^2)$. Passing to the limit $j\to\infty$ in (3.44)–(3.45), we easily obtain the integral equation for $\tilde{K}(u)$. This proves part (2). We now proceed to part (3). From an estimate similar to (3.91), we obtain

$$\sup_{|t|\leqslant T} t^2\,|(V_2 * (\operatorname{Im}(\overline{u_j}\,Ju_j)), \operatorname{Im}(\overline{u_j}\,Ju_j))$$
$$-(V_2 * (\operatorname{Im}(\bar{u}Ju)), \operatorname{Im}(\bar{u}Ju))|\to 0 \qquad \text{as} \quad j\to\infty,$$

for any $T>0$. Therefore, $\beta_j\to\beta$ in $C(\mathbb{R};\mathbb{R})$ as $j\to\infty$. We next prove that the integrands on the R.H.S. of (3.100) are locally integrable on $\mathbb{R}$. It suffices to consider the last integral. Since the map $t\mapsto |Ju(t)|^2+2t^2(V_1+V_2*|u|^2)\,|u(t)|^2$ is continuous from $\mathbb{R}$ to L^1, it is enough to show that $V_2 * ((\bar{u}\tilde{K}(u))\in L^1_{\text{loc}}(\mathbb{R}; L^\infty)$. This fact follows from the estimate

$$\int_{-T}^{T}\|V_2 * (\bar{u}\tilde{K}(u))\|_\infty\,ds$$
$$\leqslant C\int_{-T}^{T}(\|u\|\;\|\tilde{K}(u)\|+\|u\|_q\;\|\tilde{K}(u)\|_q)\,ds$$
$$\leqslant C(T+T^{1-\delta})\;|||u|||_{X(T;q)}\;|||\tilde{K}(u)|||_{X(T;q)}, \qquad T>0,\, q\in\Gamma.$$

It remains to prove (3.100). This identity follows from (3.87) by a limiting argument as in (3.80)–(3.83) with some modifications. It should be noted here that $V_2 * (\overline{u_j}\,Ku_j)\to V_2 * (\bar{u}\tilde{K}(u))$ in $L^1_{\text{loc}}(\mathbb{R}; L^\infty)$ as $j\to\infty$. Q.E.D.

Proof of Theorem 2. It remains to prove part (3). We first prove that $u\in C(\mathbb{R}\backslash\{0\};\, H^{1,-1})$ and u has the estimate (1.5). From the relation $\nabla=(1/it)(J-x)$, it follows that

$$\|u(t)\|_{1,-1}\leqslant C\,\|(1+|x|)^{-1}\,\nabla u(t)\|+C\,\|(1+|x|)^{-1}\,u(t)\|$$
$$\leqslant C\,|t|^{-1}\,(\|Ju(t)\|+\|u(t)\|)+C\,\|u(t)\|$$
$$\leqslant C(\|\phi\|_{0,1})(|t|^{-1}+1), \qquad t\in\mathbb{R}\backslash\{0\},$$

where we have used (3.73). Similarly,

$$\|u(t)-u(\tau)\|_{1,-1}\leqslant C\,\|t^{-1}Ju(t)-\tau^{-1}J(\tau)\|+C\,\|t^{-1}u(t)-\tau^{-1}u(\tau)\|$$
$$+C\,\|u(t)-u(\tau)\|, \qquad t,\tau\in\mathbb{R}\backslash\{0\},$$

from which we obtain $u\in C(\mathbb{R}\backslash\{0\};\, H^{1,-1})$ and $\Delta u\in C(\mathbb{R}\backslash\{0\};\, H^{-1,-1})$. From the relation $t\,\nabla u(t)=-i(Ju(t)-x\phi)+ix(u(t)-\phi)$, we have $tu(t)\to 0$

in $H^{-1,-1}$ as $t \to \pm 0$, so that $tu \in C(\mathbb{R}; H^{1,-1})$. We next prove that $F(u) \in C(\mathbb{R}\backslash\{0\}; H^{-1,0})$. Let $q \in \Gamma$ and $\psi \in H^{1,0}$. By Lemma 2.4, we have

$$|(V_1 u(t), \psi)| \leqslant C \cdot (\|\phi\| + \|u(t)\|_q)\, \|\psi\|_{1,0}, \qquad t \in \mathbb{R}\backslash\{0\},$$

$$\begin{aligned}|(V_1(u(t)-u(\tau)), \psi)| \leqslant C(&\|u(t)-u(\tau)\| \\ &+ \|u(t)-u(\tau)\|_q)\, \|\psi\|_{1,0}, \qquad t, \tau \in \mathbb{R}\backslash\{0\}.\end{aligned}$$

By Lemmas 2.1–2.2, we have

$$\begin{aligned}&|((V_2 * |u|^2)\, u(t), \psi)| \\ &\quad \leqslant C\, \|\psi\|\, \|\phi\| \sum_{k=2}^{3} \|u(t)\|^2_{2n/(n-\gamma_k)}, \quad t \in \mathbb{R}\backslash\{0\}, \\ &|((V_2 * |u|^2)\, u(t) - (V_2 * |u|^2)\, u(\tau), \psi)| \\ &\quad \leqslant C\, \|\psi\|\, \|u(t)-u(\tau)\| \\ &\qquad \times \sum_{k=2}^{3} (\|u(t)\|^2_{2n/(n-\gamma_k)} + \|u(\tau)\|^2_{2n/(n-\gamma_k)}), \quad t, \tau \in \mathbb{R}\backslash\{0\}.\end{aligned}$$

From these estimates and Lemma 2.11 we deduce $F(u) \in C(\mathbb{R}\backslash\{0\}; H^{-1,0})$. Hence u satisfies (1.3) and the differential equation (**) in $C(\mathbb{R}\backslash\{0\}; \bigcap_{j=0}^{1} H^{j-2,-j})$. If $\lambda_2 = \lambda_3 = 0$, then the required estimate follows in the same way as in the proof of (1.5). Q.E.D.

Proof of Corollary 1.1. The result follows from Lemma 2.11 and part (1) of Theorem 2. Q.E.D.

Proof of Theorem 3. By virtue of Lemmas 2.10 and 3.2, the proof is analogous to that of Theorem 1. Q.E.D.

Proof of Theorem 4. It remains to prove part (4). We first prove that $u \in C(\mathbb{R}\backslash\{0\}; H^{2,-2})$ and u has the estimate (1.10). From the relation $t^2\Delta = |x|^2 - 2tA - J^2 = -|x|^2 + 2x \cdot J + nit - J^2$, it follows that

$$\begin{aligned}\|u(t)\|_{2,-2} &\leqslant C\, \|(1+|x|)^{-2}\, \Delta u(t)\| + C\, \|(1+|x|)^{-2}\, u(t)\| \\ &\leqslant C(|t|^{-2} + |t|^{-1} + 1)\, \|\phi\| \\ &\quad + C\, |t|^{-2}\, (\|Ju(t)\| + \|J^2 u(t)\|) \\ &\leqslant C(\|\phi\|_{0,2})(|t|^{-2} + |t|^{2/(2-\gamma^0)}), \quad t \in \mathbb{R}\backslash\{0\},\end{aligned}$$

where we have used (3.73) and (3.99). Similarly,

$$\begin{aligned}\|u(t)-u(\tau)\|_{2,-2} \leqslant\ & C(|t|^{-2}+1)\,\|u(t)-u(\tau)\| \\ &+C\,\|t^{-1}u(t)-\tau^{-1}u(\tau)\| \\ &+C\,\|t^{-2}J^2u(t)-\tau^{-2}J^2u(\tau)\| \\ &+C\,\|t^{-2}Ju(t)-\tau^{-2}Ju(\tau)\|, \qquad t,\tau\in\mathbb{R}\backslash\{0\},\end{aligned}$$

from which we obtain $u\in C(\mathbb{R}\backslash\{0\};\ H^{2,-2})$ and $\Delta u\in C(\mathbb{R}\backslash\{0\};H^{0,-2})$. From the relation $t^2\Delta u(t)=|x|^2(\phi-u(t))+2x\cdot(Ju(t)-x\phi)+nitu(t)+(|x|^2\phi-J^2u(t))$ and (1.4), we have $t^2u(t)\to 0$ in $H^{2,-2}$ as $t\to\pm 0$, so that $t^2u\in C(\mathbb{R};H^{2,-2})$. Since we already know $t^2F(u)\in C(\mathbb{R};L^2)$, we conclude by part (3) of Theorem 2 that $u\in C^1(\mathbb{R}\backslash\{0\};H^{2,-2})$ and that u satisfies the differential equation (**) in $C(\mathbb{R}\backslash\{0\};\ \bigcap_{j=0}^2 H^{j,-j})$. Then by (**), we obtain $t^2\,\partial_t u\in C(\mathbb{R};\ \bigcap_{j=0}^2 H^{j-2,-j})$ and therefore the relation $\tilde{K}(u)=J^2u+2t^2F(u)$ reduces to (1.9). The proof of (1.11) is similar to that of (1.10). Q.E.D.

Proof of Corollary 1.2. The result follows from Lemma 2.11 and part (1) of Theorem 4. Q.E.D.

Proof of Theorem 5. From Lemma 3.4 it follows that $J^\alpha u\in\mathscr{X}$ for all $|\alpha|\leqslant k$. In order to show $u\in C(\mathbb{R}\backslash\{0\};\ \bigcap_{j=0}^k H^{j,-j})$, we define the differential operators $P_{m,j}$ ($m\in\mathrm{N}\cup\{0\}$, $j=1,...,n$) by

$$P_{m,j}=\sum_{l=0}^{m}\binom{m}{l}(-x_j)^{m-l}J_j^l \qquad (m\in\mathbb{N});\ P_{0,j}=\mathbb{1}.$$

Note that

$$(J_j-x_j)P_{m,j}=P_{m+1,j}-mitP_{m-1,j}, \qquad m\in\mathbb{N}.$$

Using this relation, we have

$$(J_j-x_j)^{2m}=\sum_{l=0}^{m}a_{m,l}t^{m-l}P_{2l,j}, \qquad m\in\mathbb{N}\cup\{0\}, \tag{3.101}$$

$$(J_j-x_j)^{2m+1}=\sum_{l=0}^{m}b_{m,l}t^{m-l}P_{2l+1,j}, \qquad m\in\mathbb{N}\cup\{0\}, \tag{3.102}$$

for some constants $a_{m,l}$, $b_{m,l}$. Since $\partial^\alpha=(it)^{-|\alpha|}\prod_{j=1}^n(J_j-x_j)^{\alpha_j}$ for $t\in\mathbb{R}\backslash\{0\}$, we have the assertion by the continuity of $J^\alpha u$ ($|\alpha|\leqslant k$) with the expressions (3.101)–(3.102). Q.E.D.

Proof of Corollary 1.3. The result follows from Lemma 2.11 and Theorem 5. Q.E.D.

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